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Abstract The interior transmission problem (ITP), which plays a fundamental role in inverse scattering theories involving penetrable defects, is investigated within the framework of mechanical waves scattered by piecewise-homogeneous, elastic or viscoelastic obstacles in a likewise heterogeneous background solid. For generality, the obstacle is allowed to be multiply connected, having both penetrable components (inclusions) and impenetrable parts (cavities). A variational formulation is employed to establish *sufficient* conditions for the existence and uniqueness of a solution to the ITP, provided that the excitation frequency does not belong to (at most) countable spectrum of transmission eigenvalues. The featured sufficient conditions, expressed in terms of the mass density and elasticity parameters of the problem, represent an advancement over earlier works on the subject in that i) they pose a precise, previously unavailable provision for the well-posedness of the ITP in situations when both the obstacle and the background solid are heterogeneous, and ii) they are dimensionally consistent i.e. invariant under the choice of physical units. For the case of a viscoelastic scatterer in an elastic solid it is further shown, consistent with earlier studies in acoustics, electromagnetism, and elasticity that the uniqueness of a solution to the ITP is maintained irrespective of the vibration frequency. When applied to the situation where *both* the scatterer and the background medium are viscoelastic i.e. dissipative, on the other hand, the same type of analysis shows that the analogous claim of uniqueness does not hold. Physically, such anomalous behavior of the "viscoelastic-viscoelastic" case (that has eluded previous studies) has its origins in a lesser known fact that the homogeneous ITP is not mechanically insulated from its surroundings – a feature that is particularly cloaked in situations when either the background medium or the scatterer are dissipative. A set of numerical results, computed for ITP configurations that meet the sufficient conditions for the existence of a solution, is included to illustrate the problem. Consistent with the preceding analysis, the results indicate that the set of transmission values is indeed empty in the "elastic-viscoelastic" case, and countable for "elastic-elastic" and "viscoelastic-viscoelastic" configurations.

Keywords interior transmission problem · transmission eigenvalues · piecewise-homogeneous media · anisotropic viscoelasticity · existence · uniqueness

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1 Introduction

Over the past two decades mathematical theories of inverse scattering have, to a large degree, experienced a paradigm shift, most notably through the development of the so-called qualitative or sampling methods [5] for non-iterative obstacle reconstruction from remote measurements of the scattered field. These techniques, which provide a powerful alternative to the customary minimization approaches and weak-scatterer approximations, are commonly centered around the development of an indicator function, that varies with coordinates of the interior sampling point, and projects remote observations of the scattered field onto a suitable functional space synthesizing the "baseline" wave motion inside the background (i.e. obstacle-free) domain. Such indicator function is normally designed to reach extreme values when the sampling point "falls" inside the support of the hidden scatterer, thereby providing an computationally-effective platform for geometric obstacle reconstruction. Among the diverse field of sampling methods [41,18], one may mention the *linear sampling method* [15,14] and the *factorization method* [27] among the most prominent examples. In the context of penetrable scatterers (e.g. elastic inclusions within the framework of mechanical waves), these theories have exposed the need to study and understand a non-traditional boundary value problem, termed the *interior transmission problem* (ITP), where two bodies with common support are subjected to a prescribed jump in Cauchy data between their boundaries. Covered by no classical theory, this problem has been the subject of early investigations since late 1980's [16,44]. The critical step in studying the ITP involves determination of conditions (in terms of input parameters) under which the problem is well-posed in the sense of Hadamard. Invariably, this leads to the analysis of the interior transmission eigenvalues, i.e. frequencies for which the homogeneous ITP permits a non-trivial solution. In particular, the characterization of such eigenvalue set has become of key importance in recent studies [40,26].

So far, two distinct methodologies have been pursued to investigate the well-posedness of the ITP, mainly within the context of Helmholtz and Maxwell equations. On the one hand, integral equation-type formulations have been developed in [16,44] for scalar-wave problems, and later adapted to deal with electromagnetic waves [23,25]. One the other hand, starting from the seminal work in [24], an alternative treatment of the ITP has been developed in [6] that involves a customized variational formulation combined with the compact perturbation argument. This approach has since been successfully applied in a series of papers to a variety of acoustic and electromagnetic scattering problems, see e.g. [7,8].

In the context of elastic waves, investigation of the ITP has been spurred by the introduction of the linear sampling method for far-field [1,12] and near-field [39,2,38,22] inverse scattering problems, as well as the development of the factorization method for elastodynamics [13]. To date, the elastodynamic ITP has been investigated mainly within the framework established for the Helmholtz and Maxwell equations, notably via integral equation approach [12] for homogeneous dissipative scatterers, and the variational treatment [10] for heterogeneous, anisotropic, and elastic scatterers in a homogeneous elastic background. Recently, a method combining integral equation approach and compact perturbation argument has been proposed in [11] for homogeneous-isotropic elasticity to obtain sufficient conditions for the well-posedness of the ITP.

To extend the validity of the linear sampling and factorization methods to a wider and more realistic class of inverse scattering problems, the focus of this study is the ITP for situations where both the obstacle and the background solid are piecewise-homogeneous, anisotropic, and either elastic or viscoelastic. This type of heterogeneity concerning the background solid has particular relevance to e.g. seismic imaging and non-destructive material testing where layered configurations are common, as created either via natural deposition or the manufacturing process. For generality, the obstacle is allowed to be multiply connected, having both penetrable components (inclusions) and impenetrable parts (cavities). In this setting, emphasis is made on the well-posendess of the visco-elastodynamic ITP, and in particular on the sufficient conditions under which the set of interior transmission eigenvalues is either countable or empty. For an in-depth study of the problem, a variational approach that generalizes upon the results in [6] and [10] is developed, including a treatment of the less-understood "viscoelastic-viscoelastic" case where both the obstacle and the background solid are dissipative. The key result of the proposed developments are the sufficient conditions under which the ITP involving piecewise-homogeneous, anisotropic, and viscoelastic solids is well-posed provided that the excitation frequency does not belong to (at most) countable spectrum of transmission eigenvalues. These conditions aim to overcome some of the limitations of the earlier treatments in (visco-) elastodynamics in that: i) they pose a precise, previously unavailable provision for the well-posedness of the ITP in situations when the obstacle and the background solid are both heterogeneous, and ii) they are dimensionally consistent i.e. invariant under the choice of physical units.

2 Preliminaries

Consider a piecewise-homogeneous, "background" viscoelastic solid $\mathit{\Omega} \subset \mathbb{R}^3$ (not necessarily bounded and isotropic) composed of N homogeneous regular regions $\mathit{\Omega}_n$. Assuming time-harmonic motion with implicit factor $e^{\mathrm{i}\omega \mathrm{t}}$ and making reference to the correspondence principle [21], let $\rho > 0$ and $\boldsymbol{\mathcal{C}}$ denote respectively the piecewise-constant mass density and (complex-valued) viscoelasticity tensor characterizing $\mathit{\Omega}$. For clarity, all quantities appearing in this study are interpreted as *dimensionless* following the scaling scheme in Table 1 where d_0 is the characteristic length, κ_0 is the reference elastic modulus, and ρ_0 is the reference mass density. Without loss of generality, ρ_0 can be taken such that $\inf\{\rho(\boldsymbol{x}) : \boldsymbol{x} \in \mathit{\Omega}\} = 1$, leaving the choice of κ_0 at this point arbitrary.

Table 1: Scaling scheme

	Dimensionless quantity	Scale
Mass density	ρ	ρ_0
Viscoelasticity tensor, traction vector	$\boldsymbol{\mathcal{C}}, \boldsymbol{t}$	κ_0
Displacement and position vectors	$\boldsymbol{u}, \boldsymbol{x}$	d_0
Vibration frequency	ω	$d_0^{-1}\sqrt{\kappa_0/\rho_0}$

Next, let $\mathit{\Omega}$ be perturbed by a bounded obstacle $\overline{D} \subset \mathit{\Omega}$ composed of M_* homogeneous viscoelastic inclusions D_*^m and $M_{\rm o}$ disconnected cavities $D_{\rm o}^j$. In this setting one may write $\overline{D} = \overline{D_* \cup D_{\rm o}}$, where $\overline{D_*} = \bigcup_{m=1}^{M_*} \overline{D_*^m}$ and $\overline{D_{\rm o}} = \bigcup_{j=1}^{M_{\rm o}} \overline{D_{\rm o}^j}$. Here it is assumed that the cavities are separated from inclusions i.e. $\overline{D_*} \cap \overline{D_{\rm o}} = \emptyset$, and that $\mathit{\Omega} \backslash \overline{D_{\rm o}}$ is connected. Similar to the case of the background solid, the viscoelasticity tensor $\boldsymbol{\mathcal{C}}_*$ and mass density $\rho_* > 0$ characterizing D_* are understood in a piecewise-constant sense. For the purpose of this study, the reference length d_0 appearing in Table 1 can be taken as $d_0 = |D|^{1/3}$, i.e. as the characteristic obstacle size.

To facilitate the ensuing discussion, consider next N_* subsets $\mathit{\Theta}_*^p$ of D_* where both $(\boldsymbol{\mathcal{C}},\rho)$ and $(\boldsymbol{\mathcal{C}}_*,\rho_*)$ are constant, i.e.

$$\forall(n,m)\in\{1,\ldots,N\}\times\{1,\ldots,M_*\}\quad \overline{\mathit{\Omega}_n}\cap\overline{D_*^m}\neq\emptyset\;\Rightarrow\;\exists p\in\{1,\ldots,N_*\}:\overline{\mathit{\Theta}_*^p}=\overline{\mathit{\Omega}_n}\cap\overline{D_*^m}.$$

Since $D_*\subset\mathit{\Omega}$, one has $M_*\leqslant N_*$ and geometrically $\overline{D_*}=\bigcup_{p=1}^{N_*}\overline{\mathit{\Theta}_*^p}$. Likewise, one may identify the N_o subsets, $\mathit{\Theta}_\mathrm{o}^q$, of $\overline{D_\mathrm{o}}$ where $(\boldsymbol{\mathcal{C}},\rho)$ is constant

$$\forall(n,j)\in\{1,\ldots,N\}\times\{1,\ldots,M_\mathrm{o}\}\quad \overline{\mathit{\Omega}_n}\cap\overline{D_\mathrm{o}^j}\neq\emptyset\;\Rightarrow\;\exists q\in\{1,\ldots,N_\mathrm{o}\}:\overline{\mathit{\Theta}_\mathrm{o}^q}=\overline{\mathit{\Omega}_n}\cap\overline{D_\mathrm{o}^j},$$

see also Fig. 2. In each $\mathit{\Theta}_*^p$, the mass density of the inclusion and the background medium will be denoted respectively by ρ_*^p and ρ^p; the background mass density in each $\mathit{\Theta}_\mathrm{o}^q$ will be similarly denoted by ρ_o^q.

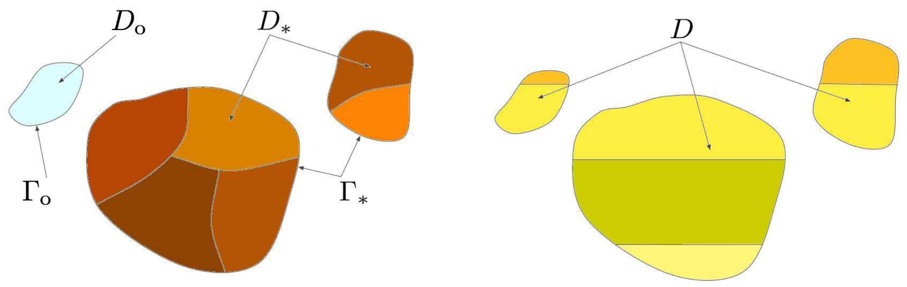

Fig. 1: ITP configuration: scatterer composed of inclusions D_* and cavities D_o (left) and scatterer support, D, occupied by the background material (right).

In what follows it is assumed that $\boldsymbol{\mathcal{C}}_*$ and $\boldsymbol{\mathcal{C}}$, synthesizing respectively the anisotropic viscoelastic behavior of the obstacle and the background, have the following properties.

Definition 1 Let $\Re[\cdot]$ and $\Im[\cdot]$ denote respectively the real and imaginary part of a complex-valued quantity. The fourth-order tensors $\boldsymbol{\mathcal{C}}$ and $\boldsymbol{\mathcal{C}}_*$ are bounded by piecewise-constant, real-valued, strictly positive functions $\mathsf{c},\mathsf{c}_*,\mathsf{C}$ and C_* and non-negative functions $\mathsf{v},\mathsf{v}_*,\mathsf{V}$ and V_* such that

$$\begin{aligned}\mathsf{c}|\boldsymbol{\xi}|^2\leqslant\Re[\boldsymbol{\xi}:\boldsymbol{\mathcal{C}}:\bar{\boldsymbol{\xi}}]\leqslant\mathsf{C}|\boldsymbol{\xi}|^2&\quad\text{in }\mathit{\Omega},\\ \mathsf{c}_*|\boldsymbol{\xi}|^2\leqslant\Re[\boldsymbol{\xi}:\boldsymbol{\mathcal{C}}_*:\bar{\boldsymbol{\xi}}]\leqslant\mathsf{C}_*|\boldsymbol{\xi}|^2&\quad\text{in }D_*,\end{aligned}\tag{1}$$

and

$$\begin{aligned}\mathsf{v}|\boldsymbol{\xi}|^2\leqslant\Im[\boldsymbol{\xi}:\boldsymbol{\mathcal{C}}:\bar{\boldsymbol{\xi}}]\leqslant\mathsf{V}|\boldsymbol{\xi}|^2&\quad\text{in }\mathit{\Omega},\\ \mathsf{v}_*|\boldsymbol{\xi}|^2\leqslant\Im[\boldsymbol{\xi}:\boldsymbol{\mathcal{C}}_*:\bar{\boldsymbol{\xi}}]\leqslant\mathsf{V}_*|\boldsymbol{\xi}|^2&\quad\text{in }D_*\end{aligned}\tag{2}$$

for all complex-valued, second-order tensor fields $\boldsymbol{\xi}$ in $\mathit{\Omega}\supset D_*$. For further reference, let $\mathsf{c}^p,\mathsf{c}_*^p,\mathsf{C}^p,\mathsf{C}_*^p,\mathsf{v}^p,\mathsf{v}_*^p,\mathsf{V}^p$ and V_*^p signify the respective (constant) values of $\mathsf{c},\mathsf{c}_*,\mathsf{C},\mathsf{C}_*,\mathsf{v},\mathsf{v}_*,\mathsf{V}$ and V_* in each $\mathit{\Theta}_*^p$, $p\in\{1,\ldots,N_*\}$, and let $\mathsf{c}_\mathrm{o}^q,\mathsf{C}_\mathrm{o}^q,\mathsf{v}_\mathrm{o}^q$ and V_o^q denote the respective values of $\mathsf{c},\mathsf{C},\mathsf{v}$ and V in each $\mathit{\Theta}_\mathrm{o}^q$, $q\in\{1,\ldots,N_\mathrm{o}\}$. With such definitions, $\mathsf{V}^p=\mathsf{v}^p\equiv 0$ and $\mathsf{V}^p\geqslant\mathsf{v}^p>0$ respectively when $\boldsymbol{\mathcal{C}}$ is elastic and viscoelastic (i.e. complex-valued) in $\mathit{\Theta}^p$, with analogous restrictions applying to the bounds on $\boldsymbol{\mathcal{C}}_*$ and $\boldsymbol{\mathcal{C}}_\mathrm{o}$. In this setting, (1) and (2) de facto require that both real and imaginary parts of a viscoelastic tensor be positive definite and bounded.

Comment. With reference to the result in [34] which establishes the major symmetry of a (tensor) relaxation function by virtue of the Onsager's reciprocity principle [45] , it follows that $\boldsymbol{\mathcal{C}}_*$ and $\boldsymbol{\mathcal{C}}$ have the usual major and minor symmetries whereby

$$\begin{aligned} &\Re[\boldsymbol{\xi}:\boldsymbol{\mathcal{C}}:\bar{\boldsymbol{\xi}}] = \boldsymbol{\xi} : \Re[\boldsymbol{\mathcal{C}}] : \bar{\boldsymbol{\xi}}, \qquad \Re[\boldsymbol{\xi}:\boldsymbol{\mathcal{C}}_*:\bar{\boldsymbol{\xi}}] = \boldsymbol{\xi} : \Re[\boldsymbol{\mathcal{C}}_*] : \bar{\boldsymbol{\xi}}, \\ &\Im[\boldsymbol{\xi}:\boldsymbol{\mathcal{C}}:\bar{\boldsymbol{\xi}}] = \boldsymbol{\xi} : \Im[\boldsymbol{\mathcal{C}}] : \bar{\boldsymbol{\xi}}, \qquad \Im[\boldsymbol{\xi}:\boldsymbol{\mathcal{C}}_*:\bar{\boldsymbol{\xi}}] = \boldsymbol{\xi} : \Im[\boldsymbol{\mathcal{C}}_*] : \bar{\boldsymbol{\xi}}. \end{aligned} \tag{3}$$

One may also note that the imposition of the upper bounds, $\mathsf{C}, \mathsf{C}_*, \mathsf{V}$ and V_* in (1) and (2) is justified by the boundedness of the moduli comprising $\boldsymbol{\mathcal{C}}$ and $\boldsymbol{\mathcal{C}}_*$, whereas $\mathsf{c}, \mathsf{c}_*, \mathsf{v}$ and v_* ensure thermomechanical stability of the system [36, 20]. These upper and lower bounds can be shown to signify the extreme eigenvalues of (the real and imaginary parts of) a fourth-order viscoelasticity tensor, defined with respect to a second-order eigentensor. Explicit treatment of such eigenvalue problems is difficult in a general anisotropic case, which may feature up to six distinct eigenvalues per real and imaginary part. In the isotropic case, however, tensors $\boldsymbol{\mathcal{C}}$ and $\boldsymbol{\mathcal{C}}_*$ can be synthesized in terms of the respective (complex) shear moduli μ and μ_*, and bulk moduli K and K_*. Under such restriction, $\boldsymbol{\mathcal{C}}$ and $\boldsymbol{\mathcal{C}}_*$ have only two distinct eigenvalues [28], given respectively by $\{2\mu, 3K\}$ and $\{2\mu_*, 3K_*\}$. Depending on the sign of the real parts of the underlying Poisson's ratios ν and ν_* [42], these moduli satisfy the relationships

$$\begin{aligned} 0 < \Re[\nu] < \tfrac{1}{2} \quad &\Rightarrow \quad \mathsf{C} = 3\Re[K] \, > \, 2\Re[\mu] = \mathsf{c}, \\ -1 < \Re[\nu] < 0 \quad &\Rightarrow \quad \mathsf{C} = 2\Re[\mu] \, > \, 3\Re[K] = \mathsf{c}, \\ 0 < \Re[\nu_*] < \tfrac{1}{2} \quad &\Rightarrow \quad \mathsf{C}_* = 3\Re[K_*] \, > \, 2\Re[\mu_*] = \mathsf{c}_*, \\ -1 < \Re[\nu_*] < 0 \quad &\Rightarrow \quad \mathsf{C}_* = 2\Re[\mu_*] \, > \, 3\Re[K_*] = \mathsf{c}_*. \end{aligned} \tag{4}$$

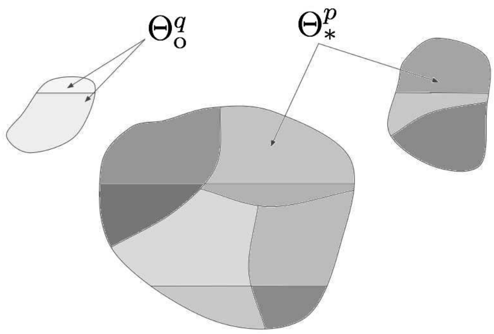

Fig. 2: Schematics of the "intersection" domains Θ_*^p and $\Theta_{\rm o}^q$ wherein the scatterer and the background solid both maintain constant material properties (see also Fig. 1).

For further reference it can be shown on the basis of (1), (2), the aforementioned eigenrepresentations of the viscoelasticity tensor, the triangle inequality, and the Cauchy-Schwarz inequality that

$$
\begin{aligned}
\left|\int_{\Theta_*^p} \boldsymbol{\xi} : \boldsymbol{\mathcal{C}}_* : \bar{\boldsymbol{\eta}} \,\mathrm{d}\boldsymbol{x}\right| &\leqslant (\mathsf{C}_* + \mathsf{V}_*)\, ||\boldsymbol{\xi}||_{L^2(\Theta_*^p)}\, ||\boldsymbol{\eta}||_{L^2(\Theta_*^p)}, \\
\left|\int_{\Theta_*^p} \boldsymbol{\xi} : \boldsymbol{\mathcal{C}} : \bar{\boldsymbol{\eta}} \,\mathrm{d}\boldsymbol{x}\right| &\leqslant (\mathsf{C} + \mathsf{V})\, ||\boldsymbol{\xi}||_{L^2(\Theta_*^p)}\, ||\boldsymbol{\eta}||_{L^2(\Theta_*^p)}, \\
\left|\int_{\Theta_{\mathrm{o}}^q} \boldsymbol{\xi} : \boldsymbol{\mathcal{C}} : \bar{\boldsymbol{\eta}} \,\mathrm{d}\boldsymbol{x}\right| &\leqslant (\mathsf{C} + \mathsf{V})\, ||\boldsymbol{\xi}||_{L^2(\Theta_{\mathrm{o}}^q)}\, ||\boldsymbol{\eta}||_{L^2(\Theta_{\mathrm{o}}^q)},
\end{aligned}
\tag{5}
$$

where $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ are square-integrable, complex-valued, second-order tensor fields in Θ_*^p and Θ_{o}^q, $p \in \{1,\ldots,N_*\}$, $q \in \{1,\ldots,N_{\mathrm{o}}\}$.

3 Interior transmission problem

Consider the time-harmonic scattering of viscoelastic waves at frequency ω where the so-called free field $\boldsymbol{u}^{\mathrm{F}}$, namely the displacement field that would have existed in the obstacle-free domain Ω, is perturbed (scattered) by a bounded obstacle $D = D_* \cup D_{\mathrm{o}} \subset \Omega$ described earlier. This boundary value problem can be conveniently written as

$$
\begin{aligned}
&\nabla\cdot[\boldsymbol{\mathcal{C}}_* : \nabla \boldsymbol{u}_*] + \rho_*\omega^2 \boldsymbol{u}_* = \boldsymbol{0} && \text{in } D_*, && \text{(6a)}\\
&\nabla\cdot[\boldsymbol{\mathcal{C}} : \nabla \boldsymbol{u}] + \rho\,\omega^2 \boldsymbol{u} = \boldsymbol{0} && \text{in } \Omega\backslash\overline{D}, && \text{(6b)}\\
&\boldsymbol{u}_* = \boldsymbol{u} + \boldsymbol{u}^{\mathrm{F}} && \text{on } \partial D_*, && \text{(6c)}\\
&\boldsymbol{t}_*[\boldsymbol{u}_*] = \boldsymbol{t}[\boldsymbol{u}] + \boldsymbol{t}[\boldsymbol{u}^{\mathrm{F}}] && \text{on } \partial D_*, && \text{(6d)}\\
&\boldsymbol{t}[\boldsymbol{u}] + \boldsymbol{t}[\boldsymbol{u}^{\mathrm{F}}] = \boldsymbol{0} && \text{on } \partial D_{\mathrm{o}} && \text{(6e)}
\end{aligned}
$$

where $\boldsymbol{u}_*$ is the (total) displacement field within piecewise-homogeneous inclusion D_*; $\boldsymbol{u}$ is the so-called scattered field signifying the *perturbation* of $\boldsymbol{u}^{\mathrm{F}}$ in $\Omega\backslash\overline{D}$ due to the presence of the scatterer; $\boldsymbol{t}_*[\boldsymbol{\vartheta}] = \boldsymbol{\mathcal{C}}_* : \nabla\boldsymbol{\vartheta}\cdot\boldsymbol{n}$ and $\boldsymbol{t}[\boldsymbol{\vartheta}] = \boldsymbol{\mathcal{C}} : \nabla\boldsymbol{\vartheta}\cdot\boldsymbol{n}$ refer to the surface tractions on ∂D; ∇ implies differentiation "to the left" [32], and $\boldsymbol{n}$ is the unit normal on the boundary of D oriented toward its exterior. Here (6a) is to be interpreted as a short-hand notation for the set of M_* governing equations applying over the respective homogeneous regions D_*^m ($m = 1, \ldots M_*$), supplemented by the continuity of displacements and tractions across ∂D_*^m where applicable. Analogous convention holds in terms of (6b) strictly applying over open homogeneous regions $\Omega_n\backslash\overline{D}$.

In what follows, it is assumed that the boundary of Ω (if any) is subject to Robin-type conditions whereby (6) are complemented by

$$
\lambda(\boldsymbol{I}_2 - \boldsymbol{N})\cdot\boldsymbol{u} + \boldsymbol{N}\cdot\boldsymbol{t}[\boldsymbol{u}] = \boldsymbol{0} \quad \text{on } \partial\Omega, \tag{7}
$$

where $\lambda > 0$ is a constant; $\boldsymbol{n}$, implicit in the definition of $\boldsymbol{t}[\boldsymbol{u}]$, is oriented outward from Ω; and $\boldsymbol{N}$ is a suitable second-order tensor that varies continuously along smooth pieces of $\partial\Omega$. Note that (7) include homogeneous Dirichlet ($\boldsymbol{N} \equiv \boldsymbol{0}$) and Neumann ($\boldsymbol{N} \equiv \boldsymbol{I}_2$) boundary conditions as special cases. In situations where Ω is unbounded (e.g. a half-space), (6) and (7) are completed by the generalized radiation condition [31], namely

$$
\lim_{R\to\infty} \int_{\Gamma_R} \Big[\boldsymbol{t}[\boldsymbol{u}](\boldsymbol{x})\cdot\mathbb{U}(\boldsymbol{x},\boldsymbol{y}) - \boldsymbol{u}(\boldsymbol{x})\cdot\mathbb{T}(\boldsymbol{x},\boldsymbol{y})\Big]\,\mathrm{d}\boldsymbol{x} \;=\; \boldsymbol{0}, \quad \forall \boldsymbol{y} \in \Omega, \tag{8}
$$

where $\Gamma_R = S_R \cap \Omega$; S_R is a sphere of radius R centered at the origin; $\mathbb{U}$ denotes the displacement Green's tensor for the obstacle-free solid Ω, and $\mathbb{T}$ is the traction Green's tensor associated with $\mathbb{U}$.

Interior transmission problem. With reference to the direct scattering framework (6)–(8), henceforth referred to as the transmission problem (TP), investigation of the associated *inverse scattering* problem in terms of the linear sampling and factorization methods [15, 12, 22, 27, 13] leads to the analysis of the so-called interior transmission problem (ITP) [5]. In the context of the present study, the ITP can be stated as the task of finding an elastodynamic field that solves the *counterpart* of (6) where the support of (6b), namely $\Omega\backslash\overline{D}$, is *replaced* by D. Previous studies have, however, shown that the analysis of an ITP is complicated by the *loss of ellipticity* relative to its "mother" TP that is well known to be elliptic. An in-depth study of this phenomenon can be found in [19] who showed, making reference to acoustic waves, that the ITP is not elliptic at any frequency. Here it is also useful to recall that the TP (6)–(8) and the associated ITP can both be represented by a common set of boundary integral equations (written over ∂D), which leads to the well-known phenomenon of fictitious frequencies [4, 30, 43] plaguing the boundary integral treatment of direct scattering problems.

For a comprehensive treatment of the problem, the ITP associated with (6)–(8) is next formulated in a general setting which i) allows for the presence of body forces, and ii) interprets the interfacial conditions over ∂D_* as a prescribed jump in Cauchy data between $\boldsymbol{u}$ and $\boldsymbol{u}_*$. Making reference to Fig. 1 and the basic concepts of functional analysis [35], such generalized ITP can be conveniently stated as a task of finding $(\boldsymbol{u}_*, \boldsymbol{u}, \boldsymbol{u}_\text{o}) \in H^1(D_*) \times H^1(D_*) \times H^1(D_\text{o})$ satisfying

$$
\begin{aligned}
&\nabla\cdot[\boldsymbol{\mathcal{C}}_*:\nabla\boldsymbol{u}_*] + \rho_*\omega^2\boldsymbol{u}_* = \boldsymbol{f}_* && \text{in } D_*, && \text{(9a)}\\
&\nabla\cdot[\boldsymbol{\mathcal{C}}:\nabla\boldsymbol{u}] + \rho\omega^2\boldsymbol{u} = \boldsymbol{f} && \text{in } D_*, && \text{(9b)}\\
&\nabla\cdot[\boldsymbol{\mathcal{C}}:\nabla\boldsymbol{u}_\text{o}] + \rho\omega^2\boldsymbol{u}_\text{o} = \boldsymbol{f} && \text{in } D_\text{o}, && \text{(9c)}\\
&\boldsymbol{u}_* = \boldsymbol{u} + \boldsymbol{g} && \text{on } \partial D_*, && \text{(9d)}\\
&\boldsymbol{t}_*[\boldsymbol{u}_*] = \boldsymbol{t}[\boldsymbol{u}] + \boldsymbol{h}_* && \text{on } \partial D_*, && \text{(9e)}\\
&\boldsymbol{t}[\boldsymbol{u}_\text{o}] = \boldsymbol{h}_\text{o} && \text{on } \partial D_\text{o}, && \text{(9f)}
\end{aligned}
$$

where $H^k \equiv W^{k,2}$ denotes the usual Sobolev space; $(\boldsymbol{f}_*, \boldsymbol{f}) \in L^2(D_*) \times L^2(D)$; $\boldsymbol{g} \in H^{\frac{1}{2}}(\partial D_*)$; $(\boldsymbol{h}_*, \boldsymbol{h}_\text{o}) \in H^{-\frac{1}{2}}(\partial D_*) \times H^{-\frac{1}{2}}(\partial D_\text{o})$, and

$$
\begin{aligned}
\boldsymbol{t}_*[\boldsymbol{u}_*] &= \boldsymbol{\mathcal{C}}_*:\nabla\boldsymbol{u}_*\cdot\boldsymbol{n} \in H^{-\frac{1}{2}}(\partial D_*),\\
\boldsymbol{t}[\boldsymbol{u}] &= \boldsymbol{\mathcal{C}}:\nabla\boldsymbol{u}\cdot\boldsymbol{n} \in H^{-\frac{1}{2}}(\partial D_*),\\
\boldsymbol{t}[\boldsymbol{u}_\text{o}] &= \boldsymbol{\mathcal{C}}:\nabla\boldsymbol{u}_\text{o}\cdot\boldsymbol{n} \in H^{-\frac{1}{2}}(\partial D_*).
\end{aligned}
\qquad (10)
$$

For completeness, it is noted that (9a)–(9c) and (9d)–(10) are interpreted respectively in the sense of distributions and the trace operator while $\boldsymbol{f}_*$ and $\boldsymbol{f}$, signifying the negatives of body forces, are placed on the right-hand side to facilitate the discussion.

Definition 2 Values of ω for which the homogeneous ITP, defined by setting $(\boldsymbol{f}_*, \boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}_*, \boldsymbol{h}_\text{o}) = (\boldsymbol{0}, \boldsymbol{0}, \boldsymbol{0}, \boldsymbol{0}, \boldsymbol{0})$ in (9), has a non-trivial solution are called *transmission eigenvalues*.

Modified interior transmission problem. To deal with anticipated non-ellipticity of the featured ITP, it is next useful to consider the compact perturbation of (9) as

$$\begin{aligned}
&\nabla\cdot[\boldsymbol{\mathcal{C}}_*:\nabla\boldsymbol{u}_*]-\rho_*\boldsymbol{u}_*=\boldsymbol{f}_* && \text{in } D_* && \text{(11a)}\\
&\nabla\cdot[\boldsymbol{\mathcal{C}}:\nabla\boldsymbol{u}]-\rho\boldsymbol{u}=\boldsymbol{f} && \text{in } D_* && \text{(11b)}\\
&\nabla\cdot[\boldsymbol{\mathcal{C}}:\nabla\boldsymbol{u}_\text{o}]-\rho\boldsymbol{u}_\text{o}=\boldsymbol{f} && \text{in } D_\text{o} && \text{(11c)}\\
&\boldsymbol{u}_*=\boldsymbol{u}+\boldsymbol{g} && \text{on } \partial D_* && \text{(11d)}\\
&\boldsymbol{t}_*[\boldsymbol{u}_*]=\boldsymbol{t}[\boldsymbol{u}]+\boldsymbol{h}_* && \text{on } \partial D_* && \text{(11e)}\\
&\boldsymbol{t}[\boldsymbol{u}_\text{o}]=\boldsymbol{h}_\text{o} && \text{on } \partial D_\text{o}, && \text{(11f)}
\end{aligned}$$

see also [6] in the context of the acoustic waves. To demonstrate the compact nature of such perturbation, one may introduce the auxiliary space

$$\begin{aligned}\varXi(D):=\big\{(\boldsymbol{u}_*,\boldsymbol{u},\boldsymbol{u}_\text{o})\in H^1(D_*)\times H^1(D_*)\times H^1(D_\text{o}):\\
\nabla\cdot[\boldsymbol{\mathcal{C}}_*:\nabla\boldsymbol{u}_*]\in L^2(D_*),\ \nabla\cdot[\boldsymbol{\mathcal{C}}:\nabla\boldsymbol{u}]\in L^2(D_*),\ \nabla\cdot[\boldsymbol{\mathcal{C}}:\nabla\boldsymbol{u}_\text{o}]\in L^2(D_\text{o})\big\},\end{aligned} \qquad (12)$$

and a differential-trace operator $\mathcal{M}$ representing (11) from $\varXi(D)$ into $L^2(D_*)\times L^2(D_*)\times L^2(D_\text{o})\times H^{\frac{1}{2}}(\partial D_*)\times H^{-\frac{1}{2}}(\partial D_*)\times H^{-\frac{1}{2}}(\partial D_\text{o})$ such that

$$\begin{aligned}\mathcal{M}(\boldsymbol{u}_*,\boldsymbol{u},\boldsymbol{u}_\text{o}):=\big(\nabla\cdot[\boldsymbol{\mathcal{C}}_*:\nabla\boldsymbol{u}_*]-\rho_*\boldsymbol{u}_*,\ \nabla\cdot[\boldsymbol{\mathcal{C}}:\nabla\boldsymbol{u}]-\rho\,\boldsymbol{u},\ \nabla\cdot[\boldsymbol{\mathcal{C}}:\nabla\boldsymbol{u}_\text{o}]-\rho\,\boldsymbol{u}_\text{o},\\
(\boldsymbol{u}_*-\boldsymbol{u})_{|\partial D_*},\ (\boldsymbol{t}_*[\boldsymbol{u}_*]-\boldsymbol{t}[\boldsymbol{u}])_{|\partial D_*},\ \boldsymbol{t}[\boldsymbol{u}_\text{o}]_{|\partial D_\text{o}}\big)\end{aligned} \qquad (13)$$

where $\boldsymbol{t}$ and $\boldsymbol{t}_*$ are defined as in (10). On the basis of (11) and (13), interior transmission problem (9) can be identified with operator $\mathcal{O}\equiv\mathcal{M}+(1+\omega^2)\mathcal{P}$ from $\varXi(D)$ into $L^2(D_*)\times L^2(D_*)\times L^2(D_\text{o})\times H^{\frac{1}{2}}(\partial D_*)\times H^{-\frac{1}{2}}(\partial D_*)\times H^{-\frac{1}{2}}(\partial D_\text{o})$, where the featured perturbation operator

$$\mathcal{P}(\boldsymbol{u}_*,\boldsymbol{u},\boldsymbol{u}_\text{o}):=\big(\rho_*\boldsymbol{u}_*,\ \rho\,\boldsymbol{u},\ \rho\,\boldsymbol{u}_\text{o},\ \boldsymbol{0},\ \boldsymbol{0},\ \boldsymbol{0}\big) \qquad (14)$$

is clearly compact by virtue of compact embedding of $H^1(D_*)$ into $L^2(D_*)$ and $H^1(D_\text{o})$ into $L^2(D_\text{o})$.

Definition 3 Triplet $(\boldsymbol{u}_*,\boldsymbol{u},\boldsymbol{u}_\text{o})\in H^1(D_*)\times H^1(D_*)\times H^1(D_\text{o})$ solving (11a)–(11c) in the sense of distributions and (11d)–(11f) in the sense of the trace operator is called a *strong solution* of the modified ITP.

3.1 Weak formulation of the modified ITP

The next step is to examine the ellipticity of the modified ITP (11) through a variational formulation, following the methodology originally introduced in [24] and later deployed in [6,10]. To this end, recall the definition of the "background" viscoelasticity tensor and consider the space of symmetric second-order tensor fields

$$W(D_*):=\Big\{\boldsymbol{\varPhi}\in L^2(D_*):\boldsymbol{\varPhi}=\boldsymbol{\varPhi}^{\text{T}},\ \nabla\cdot\boldsymbol{\varPhi}\in L^2(D_*)\ \text{ and }\ \nabla\times[\boldsymbol{\mathcal{C}}^{-1}:\boldsymbol{\varPhi}]=\boldsymbol{0}\Big\}, \qquad (15)$$

equipped with the inner product

$$(\boldsymbol{\varPhi}_1,\boldsymbol{\varPhi}_2)_{W(D_*)}:=(\boldsymbol{\varPhi}_1,\boldsymbol{\varPhi}_2)_{L^2(D_*)}+(\nabla\cdot\boldsymbol{\varPhi}_1,\nabla\cdot\boldsymbol{\varPhi}_2)_{L^2(D_*)}, \qquad (16)$$

and implied norm

$$\|\boldsymbol{\Phi}\|^2_{W(D_*)} := \|\boldsymbol{\Phi}\|^2_{L^2(D_*)} + \|\nabla\cdot\boldsymbol{\Phi}\|^2_{L^2(D_*)}. \tag{17}$$

For clarity it is noted that the curl operator in (15), defined as that "to the left" [32], is to be interpreted in the weak sense. With reference to (11) and (15), let further $\mathbb{E} := H^1(D_*) \times W(D_*) \times H^1(D_\mathrm{o})$ and define the sesquilinear form $\mathcal{A} :\, \mathbb{E}\times\mathbb{E} \to \mathbb{C}$ as

$$\begin{aligned}\mathcal{A}(U,V) :=& \int_{D_*} [\nabla \boldsymbol{u}_* : \boldsymbol{\mathcal{C}}_* : \nabla\bar{\boldsymbol{\varphi}}_* + \rho_* \boldsymbol{u}_* \cdot \bar{\boldsymbol{\varphi}}_*] \,\mathrm{d}\boldsymbol{x} + \int_{D_*} \left[\frac{1}{\rho}(\nabla\cdot\boldsymbol{\mathcal{U}})\cdot(\nabla\cdot\bar{\boldsymbol{\Phi}}) + \boldsymbol{\mathcal{U}} : \boldsymbol{\mathcal{C}}^{-1} : \bar{\boldsymbol{\Phi}}\right] \mathrm{d}\boldsymbol{x} \\ &+ \int_{D_\mathrm{o}} [\nabla \boldsymbol{u}_\mathrm{o} : \boldsymbol{\mathcal{C}} : \nabla\bar{\boldsymbol{\varphi}} + \rho\, \boldsymbol{u}_\mathrm{o} \cdot \bar{\boldsymbol{\varphi}}] \,\mathrm{d}\boldsymbol{x} - \int_{\partial D_*} \big[\boldsymbol{u}_* \cdot \bar{\boldsymbol{\Phi}} \cdot \boldsymbol{n} + (\boldsymbol{\mathcal{U}} \cdot \boldsymbol{n}) \cdot \bar{\boldsymbol{\varphi}}_*\big] \,\mathrm{d}\boldsymbol{x},\end{aligned} \tag{18}$$

together with the antilinear form $\mathcal{L} : \mathbb{E} \to \mathbb{C}$

$$\begin{aligned}\mathcal{L}(V) :=& \int_{D_*} \frac{1}{\rho} \boldsymbol{f} \cdot (\nabla\cdot\bar{\boldsymbol{\Phi}}) \,\mathrm{d}\boldsymbol{x} - \int_{D_\mathrm{o}} \boldsymbol{f} \cdot \bar{\boldsymbol{\varphi}} \,\mathrm{d}\boldsymbol{x} - \int_{D_*} \boldsymbol{f}_* \cdot \bar{\boldsymbol{\varphi}}_* \,\mathrm{d}\boldsymbol{x} \\ &+ \int_{\partial D_*} \big[\boldsymbol{h}_* \cdot \bar{\boldsymbol{\varphi}}_* - \boldsymbol{g} \cdot \bar{\boldsymbol{\Phi}} \cdot \boldsymbol{n}\big] \,\mathrm{d}\boldsymbol{x} + \int_{\partial D_\mathrm{o}} \boldsymbol{h}_\mathrm{o} \cdot \bar{\boldsymbol{\varphi}} \,\mathrm{d}\boldsymbol{x},\end{aligned} \tag{19}$$

where $\mathbb{C}$ denotes the complex plane, $U = (\boldsymbol{u}_*, \boldsymbol{\mathcal{U}}, \boldsymbol{u}_\mathrm{o}) \in \mathbb{E}$, and $V = (\boldsymbol{\varphi}_*, \boldsymbol{\Phi}, \boldsymbol{\varphi}) \in \mathbb{E}$.

With such definitions, one may recast (11) in a variational setting as a task of finding $U = (\boldsymbol{u}_*, \boldsymbol{\mathcal{U}}, \boldsymbol{u}_\mathrm{o}) \in \mathbb{E}$ such that

$$\mathcal{A}(U,V) \;=\; \mathcal{L}(V) \qquad \forall\, V = (\boldsymbol{\varphi}_*, \boldsymbol{\Phi}, \boldsymbol{\varphi}) \in \mathbb{E}. \tag{20}$$

Theorem 1 *If problem (11) has unique strong solution $(\boldsymbol{u}_*, \boldsymbol{u}, \boldsymbol{u}_o) \in H^1(D_*) \times H^1(D_*) \times H^1(D_o)$, then the variational problem (20) has unique weak solution $U = (\boldsymbol{u}_*, \boldsymbol{\mathcal{C}} : \nabla \boldsymbol{u}, \boldsymbol{u}_o) \in \mathbb{E}$. Equally, if problem (20) has unique weak solution $U = (\boldsymbol{u}_*, \boldsymbol{\mathcal{U}}, \boldsymbol{u}_o) \in \mathbb{E}$, then modified ITP (11) has unique strong solution $(\boldsymbol{u}_*, \boldsymbol{u}, \boldsymbol{u}_o) \in H^1(D_*) \times H^1(D_*) \times H^1(D_o)$ such that $(\nabla \boldsymbol{u} + \nabla^{\mathrm{T}} \boldsymbol{u})/2 = \boldsymbol{\mathcal{C}}^{-1} : \boldsymbol{\mathcal{U}}$.*

Proof The proof of this theorem has two parts. The first part establishes that $(\boldsymbol{u}_*, \boldsymbol{u}, \boldsymbol{u}_\mathrm{o})$ solves (11) if and only if $(\boldsymbol{u}_*, \boldsymbol{\mathcal{U}}, \boldsymbol{u}_\mathrm{o})$ solves (20), while the second part demonstrates the equivalence between the existence of *unique* solutions.

Parity between the existence of solutions.

- Suppose that $(\boldsymbol{u}_*, \boldsymbol{u}, \boldsymbol{u}_\mathrm{o})$ solves (11), and define $\boldsymbol{\mathcal{U}} = \boldsymbol{\mathcal{C}} : \nabla \boldsymbol{u}$ whereby $\boldsymbol{\mathcal{U}} \in W(D_*)$. By taking the $L^2(D_*)$ scalar product of (11a) with $\boldsymbol{\varphi}_* \in H^1(D_*)$ and applying the divergence theorem, one finds that

$$\begin{aligned}\int_{D_*} [\nabla \boldsymbol{u}_* : \boldsymbol{\mathcal{C}}_* : \nabla\bar{\boldsymbol{\varphi}}_* + \rho_* \boldsymbol{u}_* \cdot \bar{\boldsymbol{\varphi}}_*] \,\mathrm{d}\boldsymbol{x} - \int_{\partial D_*} (\boldsymbol{\mathcal{U}} \cdot \boldsymbol{n}) \cdot \bar{\boldsymbol{\varphi}}_* \,\mathrm{d}\boldsymbol{x} = \\ \int_{\partial D_*} \boldsymbol{h}_* \cdot \bar{\boldsymbol{\varphi}}_* \,\mathrm{d}\boldsymbol{x} - \int_{D_*} \boldsymbol{f}_* \cdot \bar{\boldsymbol{\varphi}}_* \,\mathrm{d}\boldsymbol{x},\end{aligned} \tag{21}$$

  by virtue of the boundary condition (11e). Similarly, application of the divergence theorem to the $L^2(D_\mathrm{o})$-scalar product of (11c) with $\boldsymbol{\varphi} \in H^1(D_\mathrm{o})$ yields

$$\int_{D_\mathrm{o}} [\nabla \boldsymbol{u}_\mathrm{o} : \boldsymbol{\mathcal{C}} : \nabla\bar{\boldsymbol{\varphi}} + \rho\, \boldsymbol{u}_\mathrm{o} \cdot \bar{\boldsymbol{\varphi}}] \,\mathrm{d}\boldsymbol{x} = \int_{\partial D_\mathrm{o}} \boldsymbol{h}_\mathrm{o} \cdot \bar{\boldsymbol{\varphi}} \,\mathrm{d}\boldsymbol{x} - \int_{D_\mathrm{o}} \boldsymbol{f} \cdot \bar{\boldsymbol{\varphi}} \,\mathrm{d}\boldsymbol{x}. \tag{22}$$

Finally, by taking the $L^2(D_*)$-scalar product of (11b) with $\rho^{-1}\,\nabla\cdot\boldsymbol{\Phi}$ for some $\boldsymbol{\Phi}\in W(D_*)$ and making use of (11d), one obtains

$$\begin{aligned}\int_{D_*}\left[\frac{1}{\rho}(\nabla\cdot\boldsymbol{\mathcal{U}})\cdot(\nabla\cdot\bar{\boldsymbol{\Phi}})+\boldsymbol{\mathcal{U}}:\boldsymbol{\mathcal{C}}^{-1}:\bar{\boldsymbol{\Phi}}\right]\,\mathrm{d}\boldsymbol{x}-\int_{\partial D_*}\boldsymbol{u}_*\cdot\bar{\boldsymbol{\Phi}}\cdot\boldsymbol{n}\,\mathrm{d}\boldsymbol{x}\\ =\int_{D_*}\frac{1}{\rho}\boldsymbol{f}\cdot(\nabla\cdot\bar{\boldsymbol{\Phi}})\,\mathrm{d}\boldsymbol{x}-\int_{\partial D_*}\boldsymbol{g}\cdot\bar{\boldsymbol{\Phi}}\cdot\boldsymbol{n}\,\mathrm{d}\boldsymbol{x}.\end{aligned}\tag{23}$$

The weak statement (20) is now recovered by summing (21)–(23), which demonstrates that $U=(\boldsymbol{u}_*,\boldsymbol{\mathcal{U}},\boldsymbol{u}_\mathrm{o})\in\mathbb{E}$ is indeed a solution of the variational problem.

– Conversely, let $U=(\boldsymbol{u}_*,\boldsymbol{\mathcal{U}},\boldsymbol{u}_\mathrm{o})\in\mathbb{E}$ be a weak solution to (20). Since the hypothesis $\nabla\times[\boldsymbol{\mathcal{C}}^{-1}\!:\!\boldsymbol{\mathcal{U}}]=\boldsymbol{0}$ guarantees that $\boldsymbol{\mathcal{C}}^{-1}\!:\!\boldsymbol{\mathcal{U}}$ meets the strain compatibility conditions [32], there exists a function $\boldsymbol{u}\in H^1(D_*)$ such that $(\nabla\boldsymbol{u}+\nabla^\mathrm{T}\boldsymbol{u})/2=\boldsymbol{\mathcal{C}}^{-1}:\boldsymbol{\mathcal{U}}$ in the sense of a distribution, defined up to a rigid-body motion. By virtue of the fact that U solves the variational problem (20) for all $(\boldsymbol{\varphi}_*,\boldsymbol{\Phi},\boldsymbol{\varphi})\in\mathbb{E}$, it follows by setting the triplet of weighting fields respectively to $(\boldsymbol{\varphi}_*,\boldsymbol{0},\boldsymbol{0})$, $(\boldsymbol{0},\boldsymbol{0},\boldsymbol{\varphi})$, and $(\boldsymbol{0},\boldsymbol{\Phi},\boldsymbol{0})$ that $(\boldsymbol{u}_*,\boldsymbol{u},\boldsymbol{u}_\mathrm{o})$ must be such that (21), (22) and (23) are satisfied independently.

By way of the divergence theorem, (21) yields

$$\begin{aligned}&\int_{D_*}(\nabla\cdot[\boldsymbol{\mathcal{C}}_*:\nabla\boldsymbol{u}_*]-\rho_*\boldsymbol{u}_*-\boldsymbol{f}_*)\cdot\bar{\boldsymbol{\varphi}}_*\,\mathrm{d}\boldsymbol{x}\\ &+\int_{\partial D_*}(\boldsymbol{h}_*+(\boldsymbol{\mathcal{C}}:\nabla\boldsymbol{u})\cdot\boldsymbol{n}-(\boldsymbol{\mathcal{C}}_*:\nabla\boldsymbol{u}_*)\cdot\boldsymbol{n})\cdot\bar{\boldsymbol{\varphi}}_*\,\mathrm{d}\boldsymbol{x}\;=\;0,\qquad\forall\,\boldsymbol{\varphi}_*\in H^1(D_*)\end{aligned}$$

whereby $(\boldsymbol{u}_*,\boldsymbol{u})$ satisfies

$$\begin{aligned}&\nabla\cdot[\boldsymbol{\mathcal{C}}_*:\nabla\boldsymbol{u}_*]-\rho_*\boldsymbol{u}_*=\boldsymbol{f}_*\quad&&\text{in } D_*,\\ &\boldsymbol{t}_*[\boldsymbol{u}_*]=\boldsymbol{t}[\boldsymbol{u}]+\boldsymbol{h}_*&&\text{on } \partial D_*.\end{aligned}\tag{24}$$

Similarly, equality (22) leads to

$$\int_{D_\mathrm{o}}(\nabla\cdot[\boldsymbol{\mathcal{C}}:\nabla\boldsymbol{u}_\mathrm{o}]-\rho\boldsymbol{u}_\mathrm{o}-\boldsymbol{f})\cdot\bar{\boldsymbol{\varphi}}\,\mathrm{d}\boldsymbol{x}+\int_{\partial D_\mathrm{o}}(\boldsymbol{h}_\mathrm{o}-\boldsymbol{\mathcal{C}}:\nabla\boldsymbol{u}_\mathrm{o}\cdot\boldsymbol{n})\cdot\bar{\boldsymbol{\varphi}}\,\mathrm{d}\boldsymbol{x}\;=\;0,\quad\forall\boldsymbol{\varphi}\in H^1(D_\mathrm{o})$$

which requires $(\boldsymbol{u},\boldsymbol{u}_\mathrm{o})$ to satisfy

$$\begin{aligned}&\nabla\cdot[\boldsymbol{\mathcal{C}}:\nabla\boldsymbol{u}_\mathrm{o}]-\rho\boldsymbol{u}_\mathrm{o}=\boldsymbol{f}\quad&&\text{in } D_\mathrm{o},\\ &\boldsymbol{t}[\boldsymbol{u}_\mathrm{o}]=\boldsymbol{h}_\mathrm{o}&&\text{on } \partial D_\mathrm{o}.\end{aligned}\tag{25}$$

On substituting $\boldsymbol{\mathcal{U}}=\boldsymbol{\mathcal{C}}:\nabla\boldsymbol{u}$ in (23), on the other hand, it follows that for all $\boldsymbol{\Phi}\in W(D_*)$

$$\int_{D_*}\left(\frac{1}{\rho}\nabla\cdot[\boldsymbol{\mathcal{C}}:\nabla\boldsymbol{u}]-\boldsymbol{u}-\frac{1}{\rho}\boldsymbol{f}\right)\cdot(\nabla\cdot\bar{\boldsymbol{\Phi}})\,\mathrm{d}\boldsymbol{x}+\int_{\partial D_*}(\boldsymbol{g}+\boldsymbol{u}-\boldsymbol{u}_*)\cdot\bar{\boldsymbol{\Phi}}\cdot\boldsymbol{n}\,\mathrm{d}\boldsymbol{x}\;=\;0.\tag{26}$$

To deal with (26), it is convenient to introduce the “zero-mean and zero-first-order-moment” space of vector fields

$$L^2_0(D_*)=\left\{\boldsymbol{\varphi}\in L^2(D_*):\int_{D_*}\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x}=\boldsymbol{0},\;\int_{D_*}\boldsymbol{x}\times\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x}=\boldsymbol{0}\right\},$$

and to consider solution $\boldsymbol{\chi} \in H^1(D_*)$ of the elastostatic problem

$$\begin{aligned} &\nabla\cdot[\boldsymbol{\mathcal{C}}:\nabla\boldsymbol{\chi}] = \boldsymbol{\Lambda} \quad \text{in } D_*, \qquad \boldsymbol{\Lambda} \in L^2_0(D_*), \\ &\boldsymbol{\mathcal{C}}:\nabla\boldsymbol{\chi}\cdot\boldsymbol{n} = \boldsymbol{0} \qquad \text{on } \partial D_*. \end{aligned}$$

By taking $\boldsymbol{\Phi} = \boldsymbol{\mathcal{C}}:\nabla\boldsymbol{\chi}$ in (26) whereby $\boldsymbol{\Phi} \in W(D_*)$, $\nabla\cdot\boldsymbol{\Phi} = \boldsymbol{\Lambda}$ in D_*, and $\boldsymbol{\Phi}\cdot\boldsymbol{n} = \boldsymbol{0}$ on ∂D_*, one finds that

$$\int_{D_*} \left(\frac{1}{\rho}\nabla\cdot[\boldsymbol{\mathcal{C}}:\nabla\boldsymbol{u}] - \boldsymbol{u} - \frac{1}{\rho}\boldsymbol{f}\right)\cdot\bar{\boldsymbol{\Lambda}}\,\mathrm{d}\boldsymbol{x} = 0 \qquad \forall\boldsymbol{\Lambda}\in L^2_0(D_*),$$

and consequently, using identity $(\boldsymbol{\omega}\times\boldsymbol{x})\cdot\bar{\boldsymbol{\Lambda}} = \boldsymbol{\omega}\cdot(\boldsymbol{x}\times\bar{\boldsymbol{\Lambda}})$, that

$$\frac{1}{\rho}\nabla\cdot[\boldsymbol{\mathcal{C}}:\nabla\boldsymbol{u}] - \boldsymbol{u} - \frac{1}{\rho}\boldsymbol{f} \;=\; \mathbf{c} + \boldsymbol{\omega}\times\boldsymbol{x} \qquad \text{in } D_*, \tag{27}$$

which specifies $\boldsymbol{u}$ up to an rigid-body motion given by the translation vector $\mathbf{c}$ and (infinitesimal) rotation vector $\boldsymbol{\omega}$.
Consider next solution $\boldsymbol{\chi} \in H^1(D_*)$ to the problem

$$\begin{aligned} &\nabla\cdot[\boldsymbol{\mathcal{C}}:\nabla\boldsymbol{\chi}] = \boldsymbol{0} \quad \text{in } D_* \\ &\boldsymbol{\mathcal{C}}:\nabla\boldsymbol{\chi}\cdot\boldsymbol{n} = \boldsymbol{\Lambda} \quad \text{on } \partial D_*, \qquad \boldsymbol{\Lambda}\in L^2_0(\partial D_*). \end{aligned} \tag{28}$$

Again taking $\boldsymbol{\Phi} = \boldsymbol{\mathcal{C}}:\nabla\boldsymbol{\chi}$ in (26), which this time implies $\boldsymbol{\Phi} \in W(D_*)$, $\nabla\cdot\boldsymbol{\Phi} = \boldsymbol{0}$ in D_* and $\boldsymbol{\Phi}\cdot\boldsymbol{n} = \boldsymbol{\Lambda}$ on ∂D_*, leads to

$$\int_{\partial D_*} (\boldsymbol{g} + \boldsymbol{u} - \boldsymbol{u}_*)\cdot\bar{\boldsymbol{\Lambda}}\,\mathrm{d}\boldsymbol{x} = 0 \qquad \forall\boldsymbol{\Lambda} \in L^2_0(\partial D_*), \tag{29}$$

so that

$$\boldsymbol{g} + \boldsymbol{u} - \boldsymbol{u}_* \;=\; \mathbf{c}' + \boldsymbol{\omega}'\times\boldsymbol{x} \qquad \text{on } \partial D_*, \tag{30}$$

where $\mathbf{c}'$ and $\boldsymbol{\omega}'$ are vector constants.
On substituting (27) and (30) into (26), one finds by virtue of the divergence theorem and identity $\boldsymbol{\omega}\times\boldsymbol{x} = \boldsymbol{\Omega}\cdot\boldsymbol{x}$ where $\boldsymbol{\Omega}\equiv\boldsymbol{\omega}\times\boldsymbol{I}$ that

$$\int_{\partial D_*} \big[(\mathbf{c}+\mathbf{c}')+(\boldsymbol{\omega}+\boldsymbol{\omega}')\times\boldsymbol{x}\big]\cdot\bar{\boldsymbol{\Phi}}\cdot\boldsymbol{n}\,\mathrm{d}\boldsymbol{x} + \int_{D_*} \boldsymbol{\Omega}:\boldsymbol{\Phi}\,\mathrm{d}\boldsymbol{x} \;=\; 0 \qquad \forall\boldsymbol{\Phi} \in W(D_*). \tag{31}$$

Since the second integral vanishes due to the symmetry of $\boldsymbol{\Phi}$ and antisymmetry of $\boldsymbol{\Omega}$, (31) requires that $\mathbf{c}' = -\mathbf{c}$ and $\boldsymbol{\omega} = -\boldsymbol{\omega}'$. From (24), (25), (27) and (30), it now immediately follows that $(\boldsymbol{u}_*, \boldsymbol{u} + \mathbf{c} + \boldsymbol{\omega}\times\boldsymbol{x})$ is a solution to (11).

Parity between the existence of unique solutions.

- Assume that problem (11) has a unique strong solution, and let $U^1 = (\boldsymbol{u}^1_*, \boldsymbol{\mathcal{U}}^1, \boldsymbol{u}^1_\mathrm{o})$ and $U^2 = (\boldsymbol{u}^2_*, \boldsymbol{\mathcal{U}}^2, \boldsymbol{u}^2_\mathrm{o})$ denote two weak solutions to (20). By the equivalence between solutions to the two problems, one has that $(\boldsymbol{u}^1_*, \boldsymbol{u}^1, \boldsymbol{u}^1_\mathrm{o})$ and $(\boldsymbol{u}^2_*, \boldsymbol{u}^2, \boldsymbol{u}^2_\mathrm{o})$, with $(\nabla\boldsymbol{u}^1 + \nabla^\mathsf{T}\boldsymbol{u}^1)/2 = \boldsymbol{\mathcal{C}}^{-1}:\boldsymbol{\mathcal{U}}^1$ and $(\nabla\boldsymbol{u}^2 + \nabla^\mathsf{T}\boldsymbol{u}^2)/2 = \boldsymbol{\mathcal{C}}^{-1}:\boldsymbol{\mathcal{U}}^2$, are consequently solutions to (11). Since the latter two triplets must coincide by premise, it follows that that $\boldsymbol{u}^1_* = \boldsymbol{u}^2_*$, $\boldsymbol{\mathcal{U}}^1 = \boldsymbol{\mathcal{U}}^2$ and $\boldsymbol{u}^1_\mathrm{o} = \boldsymbol{u}^2_\mathrm{o}$, i.e. that the solution to the variational problem (20) is likewise unique.

- Conversely, assume that (20) has a unique weak solution, and let $(\boldsymbol{u}^1_*, \boldsymbol{u}^1, \boldsymbol{u}^1_\mathrm{o})$ and $(\boldsymbol{u}^2_*, \boldsymbol{u}^2, \boldsymbol{u}^2_\mathrm{o})$ denote two strong solutions to (11). Since $(\boldsymbol{u}^1_*, \boldsymbol{\mathcal{C}}:\nabla\boldsymbol{u}^1, \boldsymbol{u}^1_\mathrm{o})$ and $(\boldsymbol{u}^2_*, \boldsymbol{\mathcal{C}}:\nabla\boldsymbol{u}^2, \boldsymbol{u}^2_\mathrm{o})$ are consequently solutions to (20), one must have $\boldsymbol{u}^1_* = \boldsymbol{u}^2_*$, $\nabla\boldsymbol{u}^1 + \nabla^\mathsf{T}\boldsymbol{u}^1 = \nabla\boldsymbol{u}^2 + \nabla^\mathsf{T}\boldsymbol{u}^2$ and $\boldsymbol{u}^1_\mathrm{o} = \boldsymbol{u}^2_\mathrm{o}$ by premise. The proof is completed by noting that $\boldsymbol{u}^1$ and $\boldsymbol{u}^2$ are equal up to a rigid body motion, which must vanish thanks to the boundary condition (11d).

□

4 Existence and uniqueness of a solution to the modified ITP

Having reduced the study of the modified ITP (11) to that of its variational statement (20), the question arises as to the conditions under which the latter problem is well-posed. For clarity of exposition, the focus is made on the *sufficient* conditions that compare the *elastic* parameters of the inclusion, comprising $\Re[\boldsymbol{\mathcal{C}}_*]$, to those of the background in terms of $\Re[\boldsymbol{\mathcal{C}}]$. In general, it is possible that the consideration of material dissipation (synthesized via $\Im[\boldsymbol{\mathcal{C}}_*]$ and $\Im[\boldsymbol{\mathcal{C}}]$) may relax the "elasticity" conditions under which (11) and (20) are elliptic, and thus help establish the sufficient *and* necessary conditions. The latter subject is, however, beyond the scope of this study. With such restraint, the following lemma helps establish the sufficient "elasticity" conditions.

Lemma 1 *With reference to Definition 1 specifying the bounds on the viscoelastic tensors $\boldsymbol{\mathcal{C}}$ and $\boldsymbol{\mathcal{C}}_*$, the sesquilinear form $\mathcal{A}$ is elliptic if the inequalities $\rho^p < \rho_*^p$ and $\mathsf{C}^p < \mathsf{c}_*^p$ hold in each "intersection" domain Θ_*^p, $p \in \{1,\ldots,N_*\}$.*

Proof For $U = (\boldsymbol{u}_*, \boldsymbol{\mathcal{U}}, \boldsymbol{u}_\circ) \in \mathbb{E}$, one finds from (18) that

$$\begin{aligned}\mathcal{A}(U,U) = \int_{D_*} [\nabla \boldsymbol{u}_* : \boldsymbol{\mathcal{C}}_* : \nabla \bar{\boldsymbol{u}}_* + \rho_* \boldsymbol{u}_* \cdot \bar{\boldsymbol{u}}_*] \, \mathrm{d}\boldsymbol{x} + \int_{D_*} \left[\frac{1}{\rho} (\nabla \cdot \boldsymbol{\mathcal{U}}) \cdot (\nabla \cdot \bar{\boldsymbol{\mathcal{U}}}) + \boldsymbol{\mathcal{U}} : \boldsymbol{\mathcal{C}}^{-1} : \bar{\boldsymbol{\mathcal{U}}}\right] \mathrm{d}\boldsymbol{x} \\ + \int_{D_\circ} [\nabla \boldsymbol{u}_\circ : \boldsymbol{\mathcal{C}} : \nabla \bar{\boldsymbol{u}}_\circ + \rho \, \boldsymbol{u}_\circ \cdot \bar{\boldsymbol{u}}_\circ] \, \mathrm{d}\boldsymbol{x} - \int_{\partial D_*} \left[\boldsymbol{u}_* \cdot \bar{\boldsymbol{\mathcal{U}}} \cdot \boldsymbol{n} + (\boldsymbol{\mathcal{U}} \cdot \boldsymbol{n}) \cdot \bar{\boldsymbol{u}}_*\right] \, \mathrm{d}\boldsymbol{x}.\end{aligned} \tag{32}$$

On employing the divergence theorem, the triangle inequality, the Cauchy-Schwarz inequality, and definition of the "intersection" domains Θ_*^p, one finds

$$\left| \int_{\partial D_*} \boldsymbol{\varphi}_* \cdot \bar{\boldsymbol{\Phi}} \cdot \boldsymbol{n} \, \mathrm{d}\boldsymbol{x} \right| \leqslant \sum_{p=1}^{N_*} \left[\|\boldsymbol{\varphi}_*\|_{L^2(\Theta_*^p)} \|\nabla \cdot \boldsymbol{\Phi}\|_{L^2(\Theta_*^p)} + \|\nabla \boldsymbol{\varphi}_*\|_{L^2(\Theta_*^p)} \|\boldsymbol{\Phi}\|_{L^2(\Theta_*^p)} \right]. \tag{33}$$

By virtue of the fact that $|\mathcal{A}(U,U)| \geqslant \Re[\mathcal{A}(U,U)]$, (33), and bounds (1) on (the real parts of) the viscoelasticity tensors $\boldsymbol{\mathcal{C}}_*$ and $\boldsymbol{\mathcal{C}}$ in each Θ_*^p, it can be further shown that

$$\begin{aligned}|\mathcal{A}(U,U)| \geqslant \sum_{p=1}^{N_*} \left[\mathsf{c}_*^p \|\nabla \boldsymbol{u}_*\|^2_{L^2(\Theta_*^p)} + \rho_*^p \|\boldsymbol{u}_*\|^2_{L^2(\Theta_*^p)} + \frac{1}{\rho^p} \|\nabla \cdot \boldsymbol{\mathcal{U}}\|^2_{L^2(\Theta_*^p)} + \frac{1}{\mathsf{C}^p} \|\boldsymbol{\mathcal{U}}\|^2_{L^2(\Theta_*^p)} \right] \\ - 2 \sum_{p=1}^{N_*} \left[\|\boldsymbol{u}_*\|_{L^2(\Theta_*^p)} \|\nabla \cdot \boldsymbol{\mathcal{U}}\|_{L^2(\Theta_*^p)} + \|\nabla \boldsymbol{u}_*\|_{L^2(\Theta_*^p)} \|\boldsymbol{\mathcal{U}}\|_{L^2(\Theta_*^p)} \right] \\ + \sum_{q=1}^{N_\circ} \left[\mathsf{c}_\circ^q \|\nabla \boldsymbol{u}_\circ\|^2_{L^2(\Theta_\circ^q)} + \rho_\circ^q \|\boldsymbol{u}_\circ\|^2_{L^2(\Theta_\circ^q)} \right].\end{aligned} \tag{34}$$

Since for every $(x,y) \in \mathbb{R}^2$, $\alpha > 0$, and $\beta > 0$ one has

$$\alpha x^2 + \frac{1}{\beta} y^2 - 2xy = \frac{\alpha+\beta}{2} \left(x - \frac{2}{\alpha+\beta} y \right)^2 + (\alpha - \beta) \left(\frac{1}{2} x^2 + \frac{1/\beta}{\alpha+\beta} y^2 \right), \tag{35}$$

inequality (34) can be rewritten as

$$
\begin{aligned}
|\mathcal{A}(U,U)| \geqslant \sum_{p=1}^{N_*} \Bigg[& \frac{\mathsf{c}_*^p + \mathsf{C}^p}{2} \left(\|\nabla \boldsymbol{u}_*\|_{L^2(\Theta_*^p)} - \frac{2}{\mathsf{c}_*^p + \mathsf{C}^p} \|\boldsymbol{\mathcal{U}}\|_{L^2(\Theta_*^p)} \right)^2 \\
& + (\mathsf{c}_*^p - \mathsf{C}^p) \left(\frac{1}{2} \|\nabla \boldsymbol{u}_*\|^2_{L^2(\Theta_*^p)} + \frac{1/\mathsf{C}^p}{\mathsf{c}_*^p + \mathsf{C}^p} \|\boldsymbol{\mathcal{U}}\|^2_{L^2(\Theta_*^p)} \right) \\
& + \frac{\rho_*^p + \rho^p}{2} \left(\|\boldsymbol{u}_*\|_{L^2(\Theta_*^p)} - \frac{2}{\rho_*^p + \rho^p} \|\nabla \cdot \boldsymbol{\mathcal{U}}\|_{L^2(\Theta_*^p)} \right)^2 \\
& + (\rho_*^p - \rho^p) \left(\frac{1}{2} \|\boldsymbol{u}_*\|^2_{L^2(\Theta_*^p)} + \frac{1/\rho^p}{\rho_*^p + \rho^p} \|\nabla \cdot \boldsymbol{\mathcal{U}}\|^2_{L^2(\Theta_*^p)} \right) \Bigg] \\
& + \sum_{q=1}^{N_\text{o}} \left[\mathsf{c}_\text{o}^q \|\nabla \boldsymbol{u}_\text{o}\|^2_{L^2(\Theta_\text{o}^q)} + \rho_\text{o}^q \|\boldsymbol{u}_\text{o}\|^2_{L^2(\Theta_\text{o}^q)} \right].
\end{aligned} \tag{36}
$$

On introducing the lower-bound parameter

$$
\gamma = \min_{\substack{p=1,\ldots,N_* \\ q=1,\ldots,N_\text{o}}} \left(\frac{\mathsf{c}_*^p - \mathsf{C}^p}{2}, \frac{\mathsf{c}_*^p - \mathsf{C}^p}{\mathsf{C}^p(\mathsf{c}_*^p + \mathsf{C}^p)}, \frac{\rho_*^p - \rho^p}{2}, \frac{\rho_*^p - \rho^p}{\rho^p(\rho_*^p + \rho^p)}, \mathsf{c}_\text{o}^q, \rho_\text{o}^q \right), \tag{37}
$$

one finds that $\gamma > 0$ since $\rho^p < \rho_*^p$ and $\mathsf{C}^p < \mathsf{c}_*^p$ in each Θ_*^p by premise. On the basis of this result one finds, by dropping the "squared-difference" terms in (36), that

$$
|\mathcal{A}(U,U)| \geqslant \gamma \left[\sum_{p=1}^{N_*} \left(\|\boldsymbol{u}_*\|^2_{H^1(\Theta_*^p)} + \|\boldsymbol{\mathcal{U}}\|^2_{W(\Theta_*^p)} \right) + \sum_{q=1}^{N_\text{o}} \|\boldsymbol{u}_\text{o}\|^2_{H^1(\Theta_\text{o}^q)} \right]. \tag{38}
$$

Recalling that $U = (\boldsymbol{u}_*, \boldsymbol{\mathcal{U}}, \boldsymbol{u}_\text{o}) \in \mathbb{E}$, the sesquilinear form $\mathcal{A}$ is consequently elliptic with

$$
|\mathcal{A}(U,U)| \geqslant \gamma \left(\|\boldsymbol{u}_*\|^2_{H^1(D_*)} + \|\boldsymbol{\mathcal{U}}\|^2_{W(D_*)} + \|\boldsymbol{u}_\text{o}\|^2_{H^1(D_\text{o})} \right), \tag{39}
$$

which completes the proof.

□

One is now in position to investigate the variational formulation of the modified ITP.

Theorem 2 *Under the assumptions of Lemma 1, variational problem (20) has a unique weak solution* $U = (\boldsymbol{u}_*, \boldsymbol{\mathcal{U}}, \boldsymbol{u}_\text{o}) \in \mathbb{E}$ *with an a priori estimate*

$$
\begin{aligned}
& \|\boldsymbol{u}_*\|_{H^1(D_*)} + \|\boldsymbol{\mathcal{U}}\|_{W(D_*)} + \|\boldsymbol{u}_o\|_{H^1(D_o)} \leqslant \\
& \frac{3\,C}{\gamma} \Big(\|\boldsymbol{f}_*\|_{L^2(D)} + \|\boldsymbol{f}\|_{L^2(D_*)} + \|\boldsymbol{g}\|_{H^{\frac{1}{2}}(\partial D_*)} + \|\boldsymbol{h}_*\|_{H^{-\frac{1}{2}}(\partial D_*)} + \|\boldsymbol{h}_o\|_{H^{-\frac{1}{2}}(\partial D_o)} \Big),
\end{aligned} \tag{40}
$$

where $\gamma > 0$ *is given by (37), and* $C > 0$ *is a constant independent of* $\boldsymbol{f}_*$*,* $\boldsymbol{f}$*,* $\boldsymbol{g}$*,* $\boldsymbol{h}_*$ *and* $\boldsymbol{h}_o$*.*

Proof The norm of the antilinear operator $\mathcal{L}$ in (19) can be shown, by exercising the triangle inequality, the Cauchy-Schwarz inequality, the divergence theorem (applied to $\bar{\boldsymbol{\Phi}}$) and the trace theorem, to be continuous i.e. bounded with constant $C > 0$ independent of $\boldsymbol{f}_*$, $\boldsymbol{f}$, $\boldsymbol{g}$, $\boldsymbol{h}_*$ and $\boldsymbol{h}_\text{o}$ such that

$$
\begin{aligned}
||\mathcal{L}||_{\mathbb{E}^\star} \leqslant C\Big(& \|\boldsymbol{f}_*\|_{L^2(D)} + \|\boldsymbol{f}\|_{L^2(D_*)} \\
& + \|\boldsymbol{g}\|_{H^{\frac{1}{2}}(\partial D_*)} + \|\boldsymbol{h}_*\|_{H^{-\frac{1}{2}}(\partial D_*)} + \|\boldsymbol{h}_\text{o}\|_{H^{-\frac{1}{2}}(\partial D_\text{o})} \Big),
\end{aligned} \tag{41}
$$

where $\mathbb{E}^\star$ denotes the dual of $\mathbb{E}$.

To establish the boundedness of the sesquilinear form $\mathcal{A}(U,V)$, on the other hand, one may introduce the notation

$$\begin{aligned} ||U||_{\mathbb{E}}^2 &:= \|\boldsymbol{u}_*\|^2_{H^1(D_*)} + \|\boldsymbol{\mathcal{U}}\|^2_{W(D_*)} + \|\boldsymbol{u}_\text{o}\|^2_{H^1(D_\text{o})}, \\ ||V||_{\mathbb{E}}^2 &:= \|\boldsymbol{\varphi}_*\|^2_{H^1(D_*)} + \|\boldsymbol{\Phi}\|^2_{W(D_*)} + \|\boldsymbol{\varphi}\|^2_{H^1(D_\text{o})}, \end{aligned} \tag{42}$$

for $U,V \in \mathbb{E}$ defined as in (20). In this setting, it follows from (20), the triangle inequality, (5), the Cauchy-Schwarz inequality, (33), (42), and bounds such as $||\nabla \boldsymbol{u}_*||_{L^2(D_*)} \leqslant ||U||_{\mathbb{E}}$ that there is a constant $C' > 0$ such that

$$|\mathcal{A}(U,V)| \;\leqslant\; C' \, ||U||_{\mathbb{E}} \, ||V||_{\mathbb{E}}. \tag{43}$$

Using the notation introduced in (42), (39) can also be rewritten more compactly as

$$|\mathcal{A}(U,U)| \;\geqslant\; \gamma \, ||U||^2_{\mathbb{E}}. \tag{44}$$

With the boundedness (43) and coercivity (44) of $\mathcal{A}$ now verified, the existence of a unique solution to the variational problem (20) follows directly from the Lax-Milgram theorem [35] which ensures that $||U||_{\mathbb{E}} \leqslant \gamma^{-1} ||\mathcal{L}||_{\mathbb{E}^\star}$. In this setting, a priori estimate (40) is derived as a consequence of (41), (42a), and upper bounds such as $\|\boldsymbol{u}_*\|_{H^1(D_*)} \leqslant ||U||_{\mathbb{E}}$.

□

Theorem 3 *Under the hypotheses of Lemma 1, modified ITP (11) has a unique strong solution* $(\boldsymbol{u}_*, \boldsymbol{u}, \boldsymbol{u}_o) \in H^1(D_*) \times H^1(D_*) \times H^1(D_o)$ *with an a priori estimate*

$$\begin{aligned} &\|\boldsymbol{u}_*\|_{H^1(D_*)} + \|\boldsymbol{u}\|_{H^1(D_*)} + \|\boldsymbol{u}_o\|_{H^1(D_o)} \leqslant \\ &c \left(\|\boldsymbol{f}_*\|_{L^2(D)} + \|\boldsymbol{f}\|_{L^2(D_*)} + \|\boldsymbol{g}\|_{H^{\frac{1}{2}}(\partial D_*)} + \|\boldsymbol{h}_*\|_{H^{-\frac{1}{2}}(\partial D_*)} + \|\boldsymbol{h}_o\|_{H^{-\frac{1}{2}}(\partial D_o)} \right), \end{aligned} \tag{45}$$

where $c > 0$ *is a constant independent of* $\boldsymbol{f}_*$, $\boldsymbol{f}$, $\boldsymbol{g}$, $\boldsymbol{h}_*$ *and* $\boldsymbol{h}_o$.

Proof The first part of the claim, namely the existence and uniqueness of a strong solution to (11) follow directly from Theorems 1 and 2, while inequality (45) can be obtained on the basis of (11) and (40). In particular, from the relationship $\boldsymbol{\mathcal{U}} = \boldsymbol{\mathcal{C}} : \nabla \boldsymbol{u}$ and the fact that $\boldsymbol{u}$ satisfies (11b), it follows via triangle inequality that

$$\|\boldsymbol{u}\|_{L^2(D_*)} \leqslant \alpha \left(\|\boldsymbol{\mathcal{U}}\|_{W(D_*)} + \|\boldsymbol{f}\|_{L^2(D_*)} \right), \tag{46}$$

for some constant $\alpha > 0$. By virtue of the bounds on the viscoelasticity tensor $\boldsymbol{\mathcal{C}}$ in (1) and (2), on the other hand, one finds

$$\|\nabla \boldsymbol{u}\|_{L^2(D_*)} = \|\boldsymbol{\mathcal{C}}^{-1} : \boldsymbol{\mathcal{U}}\|_{L^2(D_*)} \leqslant \beta \, \|\boldsymbol{\mathcal{U}}\|_{W(D_*)}, \tag{47}$$

for some $\beta > 0$. On combining (46) and (47) to obtain the $H^1(D_*)$ norm of $\boldsymbol{u}$, estimate (45) follows directly as a consequence of (40) with

$$c \;\leqslant\; \left(2 + \sqrt{\alpha^2 (1+\gamma)^2 + \beta^2} \right) \frac{C}{\gamma}.$$

□

5 Well-posedness of the ITP

Having established the conditions under which the modified problem (11) is uniquely solvable, one is now in position to study the existence and uniqueness of a solution to the (original) ITP (9).

Theorem 4 *Under the hypothesis that $\rho^p < \rho_*^p$ and $\mathsf{C}^p < \mathsf{c}_*^p$ in each "intersection" domain Θ_*^p, $p \in \{1,\ldots,N_*\}$ as in Lemma 1, the set of transmission eigenvalues $\omega \in \mathbb{C}$ for which the interior transmission problem (9) does not have a unique solution is either empty or forms a discrete set with infinity as the only possible accumulation point.*

Proof With reference to the space $\Xi(D)$ introduced in (12), it is recalled that the modified ITP (11) is represented by the differential-trace operator $\mathcal{M}$ as in (13), while the original problem (9) is identified with operator $\mathcal{O} = \mathcal{M} + (1+\omega^2)\mathcal{P}$, where $\mathcal{P}$ is the compact perturbation given by (14). In Theorem 3 it is shown that $\mathcal{M}^{-1}$ exists, and furthermore that it is bounded i.e. continuous under the assumptions of Lemma 1. Theorem 4 claims that the operator $\mathcal{M}+(1+\omega^2)\mathcal{P}$ is invertible for all $\omega \in \mathbb{C}\backslash S$, where S is either an empty set or a discrete set of points in the complex plane $\mathbb{C}$. Since $\mathcal{M}^{-1}$ is continuous, this claim can be established by showing the analogous result for the operator

$$\mathcal{I} \;+\; (1+\omega^2)\,\mathcal{M}^{-1}\mathcal{P},$$

where $\mathcal{I}$ is the identity operator from $\Xi(D)$ into $\Xi(D)$. As shown in Section 3, operator $\mathcal{P}$ is compact owing to the compact embedding of $H^1(D)$ into $L^2(D)$, and so is $\mathcal{M}^{-1}\mathcal{P}$ by virtue of the continuity of $\mathcal{M}^{-1}$ [35]. For this situation, the Fredholm alternative applies [47] whereby

$$\Big(\mathcal{I} + (1+\omega^2)\mathcal{M}^{-1}\mathcal{P}\Big)^{-1}$$

exists and is bounded except for, at most, a *discrete* set of transmission eigenvalues $\omega \in S \subset \mathbb{C}$ (see also Definition 2). Finally, since the countable spectrum of (compact) operator $\mathcal{M}^{-1}\mathcal{P}$ can only accumulate at zero [46], S is further characterized by infinity as the only possible accumulation point.

□

5.1 Relaxed solvability criterion

With reference to Theorem 4, it is noted that the eigenvalues of ITP (9) may form a countable set even in situations that violate the aforestated restriction: $\rho^p < \rho_*^p$ and $\mathsf{C}^p < \mathsf{c}_*^p$ in each Θ_*^p, $p \in \{1,\ldots,N_*\}$. Indeed, the latter condition can be relaxed in a way similar to that proposed in [10], albeit without introducing additional complexities. To this end, recall (9) and let $\boldsymbol{w}$ denote the "combined" elastodynamic field in $D = D_* \cup D_\mathrm{o}$ so that $\boldsymbol{u}$ and $\boldsymbol{u}_\mathrm{o}$ are the *restrictions* of $\boldsymbol{w}$ on D_* and D_o, respectively. Given $(\boldsymbol{f}_*, \boldsymbol{f}) \in L^2(D_*) \times L^2(D)$, $\boldsymbol{g} \in H^{\frac{1}{2}}(\partial D_*)$, and $(\boldsymbol{h}_*, \boldsymbol{h}_\mathrm{o}) \in H^{-\frac{1}{2}}(\partial D_*) \times H^{-\frac{1}{2}}(\partial D_\mathrm{o})$, the focus is then made on finding $(\boldsymbol{u}_*, \boldsymbol{w}) \in H^1(D_*) \times H^1(D)$ that satisfies

$$\begin{aligned}
&\nabla\cdot[\boldsymbol{\mathcal{C}}_*\!:\!\nabla\boldsymbol{u}_*] + \rho_*\omega^2\boldsymbol{u}_* = \boldsymbol{f}_* && \text{in } D_*,\\
&\nabla\cdot[\boldsymbol{\mathcal{C}}\!:\!\nabla\boldsymbol{w}] + \rho\omega^2\boldsymbol{w} = \boldsymbol{f} && \text{in } D,\\
&\boldsymbol{u}_* = \boldsymbol{w} + \boldsymbol{g} && \text{on } \partial D_*,\\
&\boldsymbol{t}_*[\boldsymbol{u}_*] = \boldsymbol{t}[\boldsymbol{w}] + \boldsymbol{h}_* && \text{on } \partial D_*,\\
&\boldsymbol{t}[\boldsymbol{w}] = \boldsymbol{h}_\mathrm{o} && \text{on } \partial D_\mathrm{o},
\end{aligned} \tag{48}$$

which is simply a restatement of (9). Following the developments in Section 3, the modified i.e. "regularized" counterpart of ITP (48) can be written as

$$\nabla\cdot[\boldsymbol{\mathcal{C}}_*:\nabla\boldsymbol{u}_*]-\rho_*\boldsymbol{u}_*=\boldsymbol{f}_* \qquad \text{in } D_*, \tag{49a}$$
$$\nabla\cdot[\boldsymbol{\mathcal{C}}:\nabla\boldsymbol{w}]-\rho\boldsymbol{w}=\boldsymbol{f} \qquad \text{in } D, \tag{49b}$$
$$\boldsymbol{u}_*=\boldsymbol{w}+\boldsymbol{g} \qquad \text{on } \partial D_*, \tag{49c}$$
$$\boldsymbol{t}_*[\boldsymbol{u}_*]=\boldsymbol{t}[\boldsymbol{w}]+\boldsymbol{h}_* \qquad \text{on } \partial D_*, \tag{49d}$$
$$\boldsymbol{t}[\boldsymbol{w}]=\boldsymbol{h}_{\rm o} \qquad \text{on } \partial D_{\rm o}, \tag{49e}$$

where $(\boldsymbol{u}_*,\boldsymbol{w})\in H^1(D_*)\times H^1(D)$. In this setting, the conditions under which the transmission eigenvalues of (9) i.e. (48) form a countable set (see Theorem 4) can be extended through the following theorem.

Theorem 5 *Under the hypothesis that $\rho^p>\rho_*^p$ and $\mathrm{c}^p>\mathsf{C}_*^p$ in each "intersection" domain Θ_*^p, $p\in\{1,\ldots,N_*\}$, the set of transmission eigenvalues $\omega\in\mathbb{C}$ for which the interior transmission problem (48) i.e. (9) does not have a unique solution is either empty or forms a discrete set with infinity as the only possible accumulation point.*

Proof The proof of the theorem follows directly from the foregoing developments provided that the variational formulation is slightly modified. To this end, define the space of second-order tensors

$$W_*(D_*):=\Big\{\boldsymbol{\Phi}_*\in L^2(D_*):\,\boldsymbol{\Phi}_*=\boldsymbol{\Phi}_*^{\rm T},\ \nabla\cdot\boldsymbol{\Phi}_*\in L^2(D_*)\text{ and }\nabla\times[\boldsymbol{\mathcal{C}}_*^{-1}:\boldsymbol{\Phi}_*]=0\Big\}, \tag{50}$$

equipped with the norm

$$\|\boldsymbol{\Phi}_*\|^2_{W_*(D_*)}:=\ \|\boldsymbol{\Phi}_*\|^2_{L^2(D_*)}+\|\nabla\cdot\boldsymbol{\Phi}_*\|^2_{L^2(D_*)}. \tag{51}$$

Note that the only difference between (15) and (50) is that $\boldsymbol{\mathcal{C}}$ has been replaced by $\boldsymbol{\mathcal{C}}_*$. Next, let $\mathbb{E}_*=W_*(D_*)\times H^1(D)$ and define the sesquilinear form $\mathcal{A}_*:\mathbb{E}_*\times\mathbb{E}_*\to\mathbb{C}$ as

$$\begin{aligned}\mathcal{A}_*(U,V)\ :=\ &\int_{D_*}\Big[\frac{1}{\rho_*}(\nabla\cdot\boldsymbol{\mathcal{U}}_*)\cdot(\nabla\cdot\bar{\boldsymbol{\Phi}}_*)+\boldsymbol{\mathcal{U}}_*:\boldsymbol{\mathcal{C}}_*^{-1}:\bar{\boldsymbol{\Phi}}_*\Big]\,\mathrm{d}\boldsymbol{x}\\&+\int_D[\nabla\boldsymbol{w}:\boldsymbol{\mathcal{C}}:\nabla\bar{\boldsymbol{\varphi}}+\rho\,\boldsymbol{w}\cdot\bar{\boldsymbol{\varphi}}]\ \mathrm{d}\boldsymbol{x}\ -\int_{\partial D_*}\big[(\boldsymbol{\mathcal{U}}_*\cdot\boldsymbol{n})\cdot\bar{\boldsymbol{\varphi}}+\boldsymbol{w}\cdot\bar{\boldsymbol{\Phi}}_*\cdot\boldsymbol{n}\big]\ \mathrm{d}\boldsymbol{x},\end{aligned} \tag{52}$$

together with the antilinear form $\mathcal{L}_*:\mathbb{E}_*\to\mathbb{C}$

$$\begin{aligned}\mathcal{L}_*(V):=&\int_{D_*}\frac{1}{\rho_*}\boldsymbol{f}_*\cdot(\nabla\cdot\bar{\boldsymbol{\Phi}}_*)\,\mathrm{d}\boldsymbol{x}-\int_D\boldsymbol{f}\cdot\bar{\boldsymbol{\varphi}}\,\mathrm{d}\boldsymbol{x}\\&+\int_{\partial D_*}\big[\boldsymbol{g}\cdot\bar{\boldsymbol{\Phi}}_*\cdot\boldsymbol{n}-\boldsymbol{h}_*\cdot\bar{\boldsymbol{\varphi}}\big]\ \mathrm{d}\boldsymbol{x}\ +\int_{\partial D_{\rm o}}\boldsymbol{h}_{\rm o}\cdot\bar{\boldsymbol{\varphi}}\,\mathrm{d}\boldsymbol{x},\end{aligned} \tag{53}$$

where $U=(\boldsymbol{\mathcal{U}}_*,\boldsymbol{w})\in\mathbb{E}_*$ and $V=(\boldsymbol{\Phi}_*,\boldsymbol{\varphi})\in\mathbb{E}_*$.

With reference to the developments in Section (3), it can be next shown that $(\boldsymbol{u}_*,\boldsymbol{w})\in H^1(D_*)\times H^1(D)$ uniquely solves ITP (49) *if and only if* $(\boldsymbol{\mathcal{U}}_*,\boldsymbol{w})\in\mathbb{E}_*$, such that $(\nabla\boldsymbol{u}_*+\nabla^{\rm T}\boldsymbol{u}_*)/2=\boldsymbol{\mathcal{C}}^{-1}:\boldsymbol{\mathcal{U}}_*$, uniquely solves the variational problem

$$\mathcal{A}_*(U,V)=\mathcal{L}_*(V)\qquad\forall V=(\boldsymbol{\Phi}_*,\boldsymbol{\varphi})\in\mathbb{E}_*. \tag{54}$$

With such equivalence, one may again make use of the fact that $|\mathcal{A}(U,U)| \geqslant \Re\,[\mathcal{A}(U,U)]$, (33), and bounds in (1) on the real parts of the viscoelasticity tensors $\boldsymbol{\mathcal{C}}_*$ and $\boldsymbol{\mathcal{C}}$ in each Θ_*^p, to show that

$$\begin{aligned} |\mathcal{A}_*(U,U)| \geqslant \sum_{p=1}^{N_*} \Big[\frac{1}{\rho_*^p} \|\nabla\cdot\boldsymbol{\mathcal{U}}_*\|^2_{L^2(\Theta_*^p)} + \frac{1}{\mathsf{C}_*^p} \|\boldsymbol{\mathcal{U}}_*\|^2_{L^2(\Theta_*^p)} + \mathsf{c}^p \|\nabla \boldsymbol{w}\|^2_{L^2(\Theta_*^p)} + \rho^p \|\boldsymbol{w}\|^2_{L^2(\Theta_*^p)} \Big] \\ - 2\sum_{p=1}^{N_*} \Big[\|\boldsymbol{w}\|_{L^2(\Theta_*^p)} \|\nabla\cdot\boldsymbol{\mathcal{U}}_*\|_{L^2(\Theta_*^p)} + \|\nabla \boldsymbol{w}\|_{L^2(\Theta_*^p)} \|\boldsymbol{\mathcal{U}}_*\|_{L^2(\Theta_*^p)} \Big] \\ + \sum_{q=1}^{N_\text{o}} \Big[\mathsf{c}_\text{o}^q \|\nabla \boldsymbol{w}\|^2_{L^2(\Theta_\text{o}^q)} + \rho_\text{o}^q \|\boldsymbol{w}\|^2_{L^2(\Theta_\text{o}^q)} \Big]. \end{aligned} \tag{55}$$

On introducing the auxiliary parameter

$$\gamma_* = \min_{\substack{p=1,\ldots,N_* \\ q=1,\ldots,N_\text{o}}} \left(\frac{\mathsf{c}^p - \mathsf{C}_*^p}{2}, \frac{\mathsf{c}^p - \mathsf{C}_*^p}{\mathsf{C}_*^p(\mathsf{c}^p + \mathsf{C}_*^p)}, \frac{\rho^p - \rho_*^p}{2}, \frac{\rho^p - \rho_*^p}{\rho_*^p(\rho^p + \rho_*^p)}, \mathsf{c}_\text{o}^q, \rho_\text{o}^q \right), \tag{56}$$

which is strictly positive ($\gamma_* > 0$) when $\rho^p > \rho_*^p$ and $\mathsf{c}^p > \mathsf{C}_*^p$ in each Θ_*^p, one finds by virtue of (35) that

$$|\mathcal{A}_*(U,U)| \geqslant \gamma_* \left[\sum_{p=1}^{N_*} \Big(\|\boldsymbol{\mathcal{U}}_*\|^2_{W_*(\Theta_*^p)} + \|\boldsymbol{w}\|^2_{H^1(\Theta_*^p)} \Big) + \sum_{q=1}^{N_\text{o}} \|\boldsymbol{w}\|^2_{H^1(\Theta_\text{o}^q)} \right]. \tag{57}$$

As a result, the sesquilinear form $\mathcal{A}_*$ is coercive with

$$|\mathcal{A}_*(U,U)| \geqslant \gamma_* \, ||U||^2_{\mathbb{E}_*}, \qquad ||U||^2_{\mathbb{E}_*} := \|\boldsymbol{\mathcal{U}}_*\|^2_{W_*(D_*)} + \|\boldsymbol{w}\|^2_{H^1(D)}. \tag{58}$$

With the continuity i.e. boundedness of both antilinear form $\mathcal{L}_*$ and sesquilinear form $\mathcal{A}_*$ being direct consequences of the triangle inequality and the Cauchy-Schwarz inequality, the hypotheses of Lax-Milgram theorem are thus verified. This in turn guarantees a unique solution to the variational problem (54) with an a priori estimate

$$\begin{aligned} \|\boldsymbol{\mathcal{U}}_*\|_{W_*(D_*)} + \|\boldsymbol{w}\|_{H^1(D)} \leqslant \frac{2\,C_*}{\gamma_*} \Big(\|\boldsymbol{f}_*\|_{L^2(D)} + \|\boldsymbol{f}\|_{L^2(D_*)} \\ + \|\boldsymbol{g}\|_{H^{\frac{1}{2}}(\partial D_*)} + \|\boldsymbol{h}_*\|_{H^{-\frac{1}{2}}(\partial D_*)} + \|\boldsymbol{h}_\text{o}\|_{H^{-\frac{1}{2}}(\partial D_\text{o})} \Big), \end{aligned} \tag{59}$$

where constant $C_* > 0$ is independent of $\boldsymbol{f}_*$, $\boldsymbol{f}$, $\boldsymbol{g}$, $\boldsymbol{h}_*$ and $\boldsymbol{h}_\text{o}$, cf. (40). Following the argument presented in Section 4, one consequently finds that the strong solution $(\boldsymbol{u}_*, \boldsymbol{w}) \in H^1(D_*) \times H^1(D)$ solving modified ITP (49) i.e. (11) is likewise unique with an estimate

$$\begin{aligned} \|\boldsymbol{u}_*\|_{H^1(D_*)} + \|\boldsymbol{w}\|_{H^1(D)} \leqslant c_* \Big(\|\boldsymbol{f}_*\|_{L^2(D)} + \|\boldsymbol{f}\|_{L^2(D_*)} \\ + \|\boldsymbol{g}\|_{H^{\frac{1}{2}}(\partial D_*)} + \|\boldsymbol{h}_*\|_{H^{-\frac{1}{2}}(\partial D_*)} + \|\boldsymbol{h}_\text{o}\|_{H^{-\frac{1}{2}}(\partial D_\text{o})} \Big), \end{aligned} \tag{60}$$

such that constant $c_* > 0$ is independent of $\boldsymbol{f}_*$, $\boldsymbol{f}$, $\boldsymbol{g}$, $\boldsymbol{h}_*$ and $\boldsymbol{h}_\text{o}$, cf. (45). The proof of Theorem 5 can be brought to a close by introducing the auxiliary space

$$\Xi_*(D) := \Big\{ (\boldsymbol{u}_*, \boldsymbol{w}) \in H^1(D_*) \times H^1(D) : \; \nabla\cdot[\boldsymbol{\mathcal{C}}_* : \nabla \boldsymbol{u}_*] \in L^2(D_*), \; \nabla\cdot[\boldsymbol{\mathcal{C}} : \nabla \boldsymbol{w}] \in L^2(D) \Big\}$$

and a *bijective* differential-trace operator $\mathcal{M}_*$, representing (49), from $\varXi_*(D)$ onto $L^2(D_*)\times L^2(D) \times H^{\frac{1}{2}}(\partial D_*) \times H^{-\frac{1}{2}}(\partial D_*) \times H^{-\frac{1}{2}}(\partial D_\text{o})$ such that

$$\mathcal{M}_*(\boldsymbol{u}_*, \boldsymbol{w}) := \Big(\nabla\!\cdot\![\boldsymbol{\mathcal{C}}_*\!:\!\nabla\boldsymbol{u}_*]-\rho_*\boldsymbol{u}_*,\ \nabla\!\cdot\![\boldsymbol{\mathcal{C}}\!:\!\nabla\boldsymbol{w}]-\rho\boldsymbol{w}, \\ (\boldsymbol{u}_*-\boldsymbol{w})_{|\partial D_*},\ (\boldsymbol{t}_*[\boldsymbol{u}_*]-\boldsymbol{t}[\boldsymbol{w}])_{|\partial D_*},\ \boldsymbol{t}[\boldsymbol{w}]_{|\partial D_\text{o}}\Big). \quad (61)$$

On defining the perturbation operator $\mathcal{P}_*$ from $\varXi_*(D)$ into $L^2(D_*)\times L^2(D)\times H^{\frac{1}{2}}(\partial D_*)\times H^{-\frac{1}{2}}(\partial D_*) \times H^{-\frac{1}{2}}(\partial D_\text{o})$, namely

$$\mathcal{P}_*(\boldsymbol{u}_*, \boldsymbol{w}) := (\rho_*\boldsymbol{u}_*,\ \rho\,\boldsymbol{w},\ \boldsymbol{0},\ \boldsymbol{0}, \boldsymbol{0}) \quad (62)$$

that is compact by virtue of compact embedding of $H^1(D_*)$ into $L^2(D_*)$ and $H^1(D)$ into $L^2(D)$, one can finally apply the Fredholm alternative to the compound operator $\mathcal{I} + (1{+}\omega^2)\,\mathcal{M}_*^{-1}\mathcal{P}_*$ whereby

$$\Big(\mathcal{I}\ +\ (1{+}\omega^2)\,\mathcal{M}_*^{-1}\mathcal{P}_*\Big)^{-1}$$

exists and is bounded except for at most a countable set of values $\omega \in S_* \subset \mathbb{C}$. Again, S_* is characterized by infinity as the only possible accumulation point, since the countable spectrum of $\mathcal{M}_*^{-1}\mathcal{P}_*$ can only accumulate at zero.

□

Remark. With reference to Theorems 4 and 5, it will be assumed throughout the remainder of this study that either

$$\rho^p < \rho^p_* \quad \text{and} \quad \mathsf{C}^p < \mathsf{c}^p_*, \qquad \forall p\in\{1,\ldots,N_*\}, \quad (63)$$

or

$$\rho^p > \rho^p_* \quad \text{and} \quad \mathsf{c}^p > \mathsf{C}^p_*, \qquad \forall p\in\{1,\ldots,N_*\}. \quad (64)$$

As shown via the foregoing theorems, the compliance with either (63) or (64) represents a *sufficient condition* for the ellipticity of the modified ITP (11) and thus for the unique solvability of ITP (9) provided that ω does not belong to a countable spectrum of transmission eigenvalues.

6 Can the set of transmission eigenvalues be empty?

In light of the foregoing results which establish sufficient conditions for the countability of the transmission eigenvalue set via the analysis of *elastic* parameters $\Re[\boldsymbol{\mathcal{C}}]$ and $\Re[\boldsymbol{\mathcal{C}}_*]$, it is next of interest to examine whether the material attenuation, manifest via $\Im[\boldsymbol{\mathcal{C}}]$ and $\Im[\boldsymbol{\mathcal{C}}_*]$, can bring about the uniqueness of a solution to the interior transmission problem (9) for all $\omega \in \mathbb{C}$. To this end, it is useful to introduce two auxiliary measures of the "viscosity" of the system

$$\mathscr{V}_{\text{min}}[\boldsymbol{\mathcal{C}}, D] \ := \ \inf\{\Im[\boldsymbol{\xi}\!:\!\boldsymbol{\mathcal{C}}\!:\!\bar{\boldsymbol{\xi}}] : \boldsymbol{x}\in D\} \ \geqslant 0, \\ \mathscr{V}_{\text{max}}[\boldsymbol{\mathcal{C}}, D] \ := \ \sup\{\Im[\boldsymbol{\xi}\!:\!\boldsymbol{\mathcal{C}}\!:\!\bar{\boldsymbol{\xi}}] : \boldsymbol{x}\in D\} \ \geqslant 0,$$

where $\boldsymbol{\xi}$ is a complex-valued, second-order tensor field in D such that $|\boldsymbol{\xi}|^2 = 1$. On the basis of Definition 1, it is clear that $\mathscr{V}_{\text{max}}[\boldsymbol{\mathcal{C}}, D]$ takes zero value only if $\Im[\boldsymbol{\mathcal{C}}]$ (and thus V) vanishes identically in D.

Theorem 6 *Let $D_o' \subseteq D_o$ and $D_*' \subseteq D_*$ denote the "viscoelastic" regions, preserving respectively the topology of D_o and D_*, that each have a support of non-zero measure. If either*

$$\mathscr{V}_{min}[\boldsymbol{\mathcal{C}}, D_o'] > 0 \qquad \textit{and} \qquad \mathscr{V}_{min}[\boldsymbol{\mathcal{C}}, D_*'] > 0 \qquad \textit{and} \qquad \mathscr{V}_{max}[\boldsymbol{\mathcal{C}}_*, D_*] = 0 \tag{65}$$

or

$$\mathscr{V}_{min}[\boldsymbol{\mathcal{C}}, D_o'] > 0 \qquad \textit{and} \qquad \mathscr{V}_{max}[\boldsymbol{\mathcal{C}}, D_*] = 0 \qquad \textit{and} \qquad \mathscr{V}_{min}[\boldsymbol{\mathcal{C}}_*, D_*'] > 0 \tag{66}$$

the interior transmission problem (9) has at most one solution. In other words, the multiplicity of solutions to ITP (9) is precluded if there is a region $D_o' \subseteq D_o$ where $\boldsymbol{\mathcal{C}}$ is viscoelastic and a region $D_*' \subseteq D_*$ where <u>either</u> $\boldsymbol{\mathcal{C}}$ or $\boldsymbol{\mathcal{C}}_*$ is viscoelastic.

Proof Let $(\boldsymbol{u}_*, \boldsymbol{u}, \boldsymbol{u}_\mathrm{o})$ be the algebraic difference between two solutions to the interior transmission problem (9). The displacement field $\boldsymbol{u}_\mathrm{o}$, being solution to the homogeneous Neumann problem over D_o, vanishes identically owing to the premise that $\mathscr{V}_{\min}[\boldsymbol{\mathcal{C}}, D_\mathrm{o}'] > 0$ where D_o' preserves the topology of D_o. From the homogeneous counterparts of (9a) and (9b), on the other hand, one finds by employing the divergence theorem together with boundary conditions $\boldsymbol{u} = \boldsymbol{u}_*$ and $\boldsymbol{t}[\boldsymbol{u}] = \boldsymbol{t}_*[\boldsymbol{u}_*]$ over ∂D_* that

$$\int_{D_*} \Big[\nabla \boldsymbol{u} : \boldsymbol{\mathcal{C}} : \nabla \bar{\boldsymbol{u}} - \rho\,\omega^2 \boldsymbol{u} \cdot \bar{\boldsymbol{u}}\Big] \,\mathrm{d}\boldsymbol{x} = \int_{\partial D_*} \boldsymbol{t}[\boldsymbol{u}] \cdot \bar{\boldsymbol{u}} \,\mathrm{d}\boldsymbol{x} =$$
$$= \int_{\partial D_*} \boldsymbol{t}_*[\boldsymbol{u}_*] \cdot \bar{\boldsymbol{u}}_* \,\mathrm{d}\boldsymbol{x} = \int_{D_*} \Big[\nabla \boldsymbol{u}_* : \boldsymbol{\mathcal{C}}_* : \nabla \bar{\boldsymbol{u}}_* - \rho_* \omega^2 \boldsymbol{u}_* \cdot \bar{\boldsymbol{u}}_*\Big] \,\mathrm{d}\boldsymbol{x}. \tag{67}$$

The triviality of $\boldsymbol{u}$ and $\boldsymbol{u}_*$ can now be established by taking the imaginary part of (67) which reads

$$\int_{D_*} \nabla \boldsymbol{u} : \Im[\boldsymbol{\mathcal{C}}] : \nabla \bar{\boldsymbol{u}} \,\mathrm{d}\boldsymbol{x} = \int_{D_*} \nabla \boldsymbol{u}_* : \Im[\boldsymbol{\mathcal{C}}_*] : \nabla \bar{\boldsymbol{u}}_* \,\mathrm{d}\boldsymbol{x}. \tag{68}$$

Assuming (65) which requires the right-hand side of (68) to vanish, one finds by virtue of (2) that

$$0 \leq \int_{D_*'} \nabla \boldsymbol{u} : \Im[\boldsymbol{\mathcal{C}}] : \nabla \bar{\boldsymbol{u}} \,\mathrm{d}\boldsymbol{x} \;\leq\; \int_{D_*} \nabla \boldsymbol{u} : \Im[\boldsymbol{\mathcal{C}}] : \nabla \bar{\boldsymbol{u}} \,\mathrm{d}\boldsymbol{x} = 0,$$

which via Korn's inequality [37,33] yields $\nabla \boldsymbol{u} = \boldsymbol{0}$ in D_*'. On recalling the field equation (9b) with $\boldsymbol{f} = \boldsymbol{0}$, it follows that $\boldsymbol{u} = \boldsymbol{0}$ in D_*' as well. By way of the Holmgren's uniqueness theorem for piecewise-homogeneous bodies [22] and hypothesis that D_*' preserves the topology of D_*, the trivial Cauchy data $\boldsymbol{u} = \boldsymbol{t}[\boldsymbol{u}] = \boldsymbol{0}$ on $\partial D_*'$ can now be uniquely extended to demonstrate that $\boldsymbol{u} = \boldsymbol{0}$ in D_* and consequently that $\boldsymbol{u} = \boldsymbol{t}[\boldsymbol{u}] = \boldsymbol{0}$ on ∂D_*. On the basis of the interfacial conditions (9d) and (9e) with $\boldsymbol{g} = \boldsymbol{0}$ and $\boldsymbol{h}_* = \boldsymbol{0}$, one further has $\boldsymbol{u}_* = \boldsymbol{t}_*[\boldsymbol{u}_*] = \boldsymbol{0}$ on ∂D_*, so that finally $\boldsymbol{u}_* = \boldsymbol{0}$ in D_* by virtue of the Holmgren's uniqueness theorem. The companion claim, namely that the solution difference $(\boldsymbol{u}_*, \boldsymbol{u}, \boldsymbol{u}_\mathrm{o})$ vanishes identically when (66) is met, can be established in an analogous fashion.

□

One is now in position to demonstrate, under suitable restriction on $\boldsymbol{\mathcal{C}}$, $\boldsymbol{\mathcal{C}}_*$, ρ and ρ_*, the existence of a unique strong solution to the interior transmission problem (9) $\forall \omega \in \mathbb{C}$.

Theorem 7 *Assuming that either (63) or (64) hold in terms of $\rho, \rho_*, \Re[\boldsymbol{\mathcal{C}}]$ and $\Re[\boldsymbol{\mathcal{C}}_*]$, and that either (65) or (66) hold in terms of $\Im[\boldsymbol{\mathcal{C}}]$ and $\Im[\boldsymbol{\mathcal{C}}_*]$, ITP (9) has a unique strong solution $(\boldsymbol{u}_*, \boldsymbol{u}, \boldsymbol{u}_o) \in H^1(D_*) \times H^1(D_*) \times H^1(D_o)$ with an a priori estimate*

$$\|\boldsymbol{u}_*\|_{H^1(D_*)} + \|\boldsymbol{u}\|_{H^1(D_*)} + \|\boldsymbol{u}_o\|_{H^1(D_o)} \;\leqslant\; c\Big(\|\boldsymbol{f}_*\|_{L^2(D)} + \|\boldsymbol{f}\|_{L^2(D_*)} \\ + \|\boldsymbol{g}\|_{H^{\frac{1}{2}}(\partial D_*)} + \|\boldsymbol{h}_*\|_{H^{-\frac{1}{2}}(\partial D_*)} + \|\boldsymbol{h}_o\|_{H^{-\frac{1}{2}}(\partial D_o)}\Big) \quad (69)$$

where constant $c\!>\!0$ is independent of $\boldsymbol{f}_*$, $\boldsymbol{f}$, $\boldsymbol{g}$, $\boldsymbol{h}_*$ and $\boldsymbol{h}_o$.

Proof The above claim is a direct consequence of Theorems 4, 5, and 6. To illustrate the proof, assume that (63) and either (65) or (66) are met, and recall the definition of operators $\mathcal{M}$ and $\mathcal{P}$ given respectively by (13) and (14). By Theorem 4, operator $\mathcal{O} = \mathcal{M} + (1\!+\!\omega^2)\mathcal{P}$ identified with ITP (9) is surjective, whereas Theorem 6 assures that $\mathcal{O}$ is injective. As a consequence, $\mathcal{O}$ is bijective with bounded inverse [5]. Thus there exists a unique solution to the interior transmission problem (9), for all $\omega \in \mathbb{C}$, verifying the a priori estimate (69). The proof when (64) holds in lieu of (63) can be established in an analogous fashion on the basis of Theorems 5 and 6, recalling that $\boldsymbol{u} \equiv \boldsymbol{w}_{|D_*}$ and $\boldsymbol{u}_{\rm o} \equiv \boldsymbol{w}_{|D_{\rm o}}$ in terms of the "combined" field $\boldsymbol{w}$ such that $(\boldsymbol{u}_*, \boldsymbol{w}) \in H^1(D_*) \times H^1(D)$ solves (48).

□

Remark. Implicit in the foregoing analysis is the fact that the solution, $\boldsymbol{u}_{\rm o}$, to the homogeneous ITP over $D_{\rm o}$ is *uncoupled* from $\boldsymbol{u}$ and $\boldsymbol{u}_*$ in that it solves the interior Neumann problem

$$\begin{aligned} &\nabla\!\cdot\![\boldsymbol{\mathcal{C}}\!:\!\nabla \boldsymbol{u}_{\rm o}] + \rho\omega^2 \boldsymbol{u}_{\rm o} = \boldsymbol{0} &&\text{in } D_{\rm o}, \\ &\boldsymbol{t}[\boldsymbol{u}_{\rm o}] = \boldsymbol{0} &&\text{on } \partial D_{\rm o}. \end{aligned}$$

As a result, $\boldsymbol{u}_{\rm o}$ will by itself introduce discrete eigenvalues into the problem [29] as soon as the restriction $\boldsymbol{\mathcal{C}}_{|D_{\rm o}}$ is elastic i.e. real-valued. This is reflected in Theorem 6 which precludes such possibility by requiring that $\mathscr{V}_{\rm min}[\boldsymbol{\mathcal{C}}, D'_{\rm o}] > 0$ where $D'_{\rm o} \subseteq D_{\rm o}$ has a support of non-zero measure and preserves the topology of $D_{\rm o}$. To provide a focus in the study, this assumption will be retained hereon.

With the above premise, consider next the "elastic-elastic" case

$$\mathscr{V}_{\rm min}[\boldsymbol{\mathcal{C}}, D'_{\rm o}] > 0 \qquad \text{and} \qquad \mathscr{V}_{\rm max}[\boldsymbol{\mathcal{C}}, D_*] = 0 \qquad \text{and} \qquad \mathscr{V}_{\rm max}[\boldsymbol{\mathcal{C}}_*, D_*] = 0,$$

where both $\boldsymbol{\mathcal{C}}$ and $\boldsymbol{\mathcal{C}}_*$ are real-valued everywhere in D_*. In this situation, both sides of (68) vanish which precludes the foregoing analysis from emptying the (countable) set of transmission eigenvalues. This is consistent with the well-known behavior of the interior Dirichlet and Neumann problems in elastodynamics [29] which are known to have discrete eigenvalues.

If the same procedure as in Theorem 6 is applied to the "viscoelastic-viscoelastic" case, on the other hand, where both $\boldsymbol{\mathcal{C}}$ and $\boldsymbol{\mathcal{C}}_*$ are (at least intermittently) complex-valued such that

$$\mathscr{V}_{\rm min}[\boldsymbol{\mathcal{C}}, D'_{\rm o}] > 0 \qquad \text{and} \qquad \mathscr{V}_{\rm min}[\boldsymbol{\mathcal{C}}, D'_*] > 0 \qquad \text{and} \qquad \mathscr{V}_{\rm min}[\boldsymbol{\mathcal{C}}_*, D''_*] > 0, \qquad (70)$$

where $D'_* \cap D_c \neq \emptyset$, $D''_* \cap D_c \neq \emptyset$, and $D_c \subset D_*$ is *connected*, one finds that both sides of (68) are non-trivial over D_c, which again fails to eliminate the transmission eigenvalues. Note that the featured assumption on D'_* and D''_* physically means that there is at least one connected piece, $D_c \subset D_*$, where *both* $\boldsymbol{\mathcal{C}}$ and $\boldsymbol{\mathcal{C}}_*$ are at least partially viscoelastic. This of course encompasses the case where $\boldsymbol{\mathcal{C}}$ and $\boldsymbol{\mathcal{C}}_*$ are complex-valued throughout. To better understand such counter-intuitive result whereby the introduction of "additional" material dissipation relative to that in Theorem 6 may lead to the loss of injectivity, it is useful to re-examine the problem within an energetic framework.

6.1 Energy balance

To establish the energetic analogue of (67) and (68), involved in the proof of Theorem 6, consider the case of steady-state viscoelastic vibrations as in [9]. With reference to the implicit time-harmonic factor $e^{\mathrm{i}\omega\mathrm{t}}$, one may recall the expressions for the *velocity* fields, $\boldsymbol{v} = \mathrm{i}\omega\boldsymbol{u}$ and $\boldsymbol{v}_* = \mathrm{i}\omega\boldsymbol{u}_*$, over D_* which allows one to interpret

$$
\begin{aligned}
\Im[\nabla\boldsymbol{u}\!:\!\boldsymbol{\mathcal{C}}\!:\!\nabla\bar{\boldsymbol{u}}] \;&=\; \frac{1}{\pi}\int_0^T \Re[\boldsymbol{\mathcal{C}}\!:\!\nabla\boldsymbol{u}\,e^{\mathrm{i}\omega\mathrm{t}}] : \Re[\nabla\boldsymbol{v}e^{\mathrm{i}\omega\mathrm{t}}]\,\mathrm{d}t \;\equiv\; \frac{1}{\pi}\,\mathcal{E}^{\mathrm{D}},\\
\Im[\nabla\boldsymbol{u}_*\!:\!\boldsymbol{\mathcal{C}}_*\!:\!\nabla\bar{\boldsymbol{u}}_*] \;&=\; \frac{1}{\pi}\int_0^T \Re[\boldsymbol{\mathcal{C}}_*\!:\!\nabla\boldsymbol{u}_*\,e^{\mathrm{i}\omega\mathrm{t}}] : \Re[\nabla\boldsymbol{v}_*e^{\mathrm{i}\omega\mathrm{t}}]\,\mathrm{d}t \;\equiv\; \frac{1}{\pi}\,\mathcal{E}_*^{\mathrm{D}},
\end{aligned}
\tag{71}
$$

in terms of the dissipated energy densities, $\mathcal{E}^{\mathrm{D}}$ and $\mathcal{E}_*^{\mathrm{D}}$ in D_*, calculated per period of vibrations $T = 2\pi/\omega$. Similarly, one finds that

$$
\begin{aligned}
\Im[\boldsymbol{t}[\boldsymbol{u}]\!\cdot\!\bar{\boldsymbol{u}}] \;&=\; \frac{1}{\pi}\int_0^T \Re[\boldsymbol{t}[\boldsymbol{u}]e^{\mathrm{i}\omega\mathrm{t}}]\cdot\Re[\boldsymbol{v}e^{\mathrm{i}\omega\mathrm{t}}]\,\mathrm{d}t \;\equiv\; \frac{1}{\pi}\,\mathcal{F}^{\mathrm{D}},\\
\Im[\boldsymbol{t}_*[\boldsymbol{u}_*]\!\cdot\!\bar{\boldsymbol{u}}_*] \;&=\; \int_0^T \Re[\boldsymbol{t}_*[\boldsymbol{u}_*]e^{\mathrm{i}\omega\mathrm{t}}]\cdot\Re[\boldsymbol{v}_*e^{\mathrm{i}\omega\mathrm{t}}]\,\mathrm{d}t \;\equiv\; \frac{1}{\pi}\,\mathcal{F}_*^{\mathrm{D}},
\end{aligned}
\tag{72}
$$

carry the meaning of energy influx densities, $\mathcal{F}^{\mathrm{D}}$ and $\mathcal{F}_*^{\mathrm{D}}$ over ∂D_*, reckoned per period of vibrations. On the basis of (71) and (72), the imaginary part of (67) can be written as

$$
\int_{D_*} \mathcal{E}^{\mathrm{D}}\,\mathrm{d}\boldsymbol{x} \;=\; \int_{\partial D_*} \mathcal{F}^{\mathrm{D}}\,\mathrm{d}\boldsymbol{x} \;\;=\;\; \int_{D_*} \mathcal{E}_*^{\mathrm{D}}\,\mathrm{d}\boldsymbol{x} \;=\; \int_{\partial D_*} \mathcal{F}_*^{\mathrm{D}}\,\mathrm{d}\boldsymbol{x}, \tag{73}
$$

which states that any solution to the homogeneous ITP must be such that the dissipated energies over D_*, and corresponding energy influxes over ∂D_*, are the same for both bodies. In this setting it is clear that when either body is purely elastic over D_* as specified by (65) and (66), the equality of dissipated energies (73) requires the displacement field in the viscoelastic companion to vanish by virtue of the positive definiteness (2) of the imaginary part of the viscoelastic tensor. From the vanishing Cauchy data on ∂D_*, one consequently finds by virtue of the Holmgren's uniqueness theorem [22] that the solution in the elastic body must vanish as well. When both bodies are viscoelastic as in (70), on the other hand, one finds from (73) that

$$
\int_{D_c} \mathcal{E}^{\mathrm{D}}\,\mathrm{d}\boldsymbol{x} \;=\; \int_{\partial D_c} \mathcal{F}^{\mathrm{D}}\,\mathrm{d}\boldsymbol{x} \;\;=\;\; \int_{D_c} \mathcal{E}_*^{\mathrm{D}}\,\mathrm{d}\boldsymbol{x} \;=\; \int_{\partial D_c} \mathcal{F}_*^{\mathrm{D}}\,\mathrm{d}\boldsymbol{x} \;>0, \tag{74}
$$

where D_c is a connected piece of D_*, and the foregoing approach provides no means to preclude the existence of non-trivial solutions to the homogeneous ITP. In particular, (74) demonstrates the homogeneous ITP is *not mechanically isolated* from its surroundings in the sense that it permits positive energy influx into both bodies over $\partial D_c \subset \partial D_*$ even though the jump between the respective Cauchy data, specified via $\boldsymbol{g}$ and $\boldsymbol{h}_*$, vanishes.

7 Results and discussion

Comparison with existing results. In Section 5, it is shown that ITP (9) is well-posed when ω does not belong to (at most) countable set of transmission eigenvalues, *provided* that either (63) or (64) holds. These sufficient conditions, formulated in terms of the material-parameter distributions $(\boldsymbol{\mathcal{C}}, \rho)$ and $(\boldsymbol{\mathcal{C}}_*, \rho_*)$, state that

$$\text{either} \qquad \rho^p < \rho_*^p,\;\; \mathsf{C}^p < \mathsf{c}_*^p \qquad \text{or} \qquad \rho^p > \rho_*^p,\;\; \mathsf{c}^p > \mathsf{C}_*^p \qquad \forall p \in \{1,\ldots,N_*\}, \tag{75}$$

where C and c signify respectively the maximum and minimum eigenvalues of the real part of a fourth-order viscoelasticity tensor $\boldsymbol{\mathcal{C}}$ as examined earlier.

To the authors' knowledge, the first (and only existing) study of an elastodynamic ITP involving heterogeneous bodies can be found in [10], who assumed that: i) the obstacle and the background are both non-dissipative i.e. elastic; ii) the background is homogeneous with unit mass density, and iii) the obstacle is in the form of a single connected inclusion with bounded but otherwise arbitrary distribution of elastic properties. Within the framework of the present investigation, these hypotheses can be summarized as

$$\Im[\boldsymbol{\mathcal{C}}] = \Im[\boldsymbol{\mathcal{C}}_*] = \boldsymbol{0}, \qquad \boldsymbol{\mathcal{C}} = \text{const.}, \qquad \rho = 1, \qquad \boldsymbol{\mathcal{C}}_* < \boldsymbol{\infty}, \qquad D \equiv D_*. \tag{76}$$

With such assumptions, [10] employed the variational formulation analogous to that in this study (following [24, 6]) and obtained sufficient conditions for the countability of the transmission eigenvalue spectrum as

$$\text{either} \qquad \rho_*^{\min} \geqslant \mathsf{c}_*^{\min} > \frac{\mathsf{C}^2}{\mathsf{c}} \qquad \text{or} \qquad \rho_*^{\max} < \frac{\mathsf{c}}{\mathsf{C}^2},\;\; \mathsf{C}_*^{\max} < \frac{\mathsf{c}}{\mathsf{C}^2}, \tag{77}$$

where

$$\begin{aligned} \rho_*^{\min} &= \inf\{\rho_* : \boldsymbol{x} \in D\}, \qquad & \rho_*^{\max} &= \sup\{\rho_* : \boldsymbol{x} \in D\}, \\ \mathsf{c}_*^{\min} &= \inf\{\mathsf{c}_* : \boldsymbol{x} \in D\}, \qquad & \mathsf{C}_*^{\max} &= \sup\{\mathsf{C}_* : \boldsymbol{x} \in D\}. \end{aligned} \tag{78}$$

Despite the fact that all quantities in (77) are dimensionless, conditions (77) are unfortunately non-informative as either set of inequalities could be, for a given ITP, *both* met and violated depending on the choice of the reference modulus κ_0 in Table 1 used to normalize $\boldsymbol{\mathcal{C}}$ and $\boldsymbol{\mathcal{C}}_*$ (note that ρ_0 must equal the mass density of the background solid to have $\rho = 1$). As a point of reference, sufficient conditions (75) obtained in this study can be degenerated by virtue of (76) and (78) to conform with the hypotheses made in [10] as

$$\text{either} \qquad \rho_*^{\min} > 1,\;\; \mathsf{c}_*^{\min} > \mathsf{C} \qquad \text{or} \qquad \rho_*^{\max} < 1,\;\; \mathsf{C}_*^{\max} < \mathsf{c}. \tag{79}$$

This counterpart of (77), that is invariant under the choice of ρ_0 and κ_0, can be qualitatively described as a requirement that the inclusion be either "denser and stiffer" or "lighter and softer" than the background solid throughout – a condition which guarantees that ITP (9), subject to hypotheses (76), is characterized by a countable spectrum of transmission eigenvalues.

In the context of dissipative solids, [12] considered the ITP for a homogeneous viscoelastic obstacle in a homogeneous elastic background. For the particular case where the prescribed jump in Cauchy data, manifest via $\boldsymbol{g}$ and $\boldsymbol{h}_*$ in the present study, is given by the traces of the elastodynamic fundamental solution, they established the existence and uniqueness of a solution to the featured ITP via a volume integral approach. Most recently, [11] investigated the ITP in isotropic elasticity for the canonical case where both the inclusion and the background solid are homogeneous. By making recourse to the integral equation approach, ellipticity of the elastostatic ITP, and the compact perturbation argument, they arrived at sufficient conditions for the countability of the transmission eigenvalue spectrum as

$$\text{either} \qquad \mu_* > \mu, \;\; K_* > K \qquad \text{or} \qquad \mu_* < \mu, \;\; K_* < K. \tag{80}$$

For completeness, sufficient conditions (75) can be degenerated by virtue of (4) to the homogeneous-isotropic-elastic case as

$$\begin{aligned} 0 < \nu < \tfrac{1}{2} \quad &\Rightarrow \quad \text{either} \quad \rho_* > \rho, \;\; 2\mu_* > 3K \quad \text{or} \quad \rho_* < \rho, \;\; 3K_* < 2\mu, \\ -1 < \nu_* < 0 \quad &\Rightarrow \quad \text{either} \quad \rho_* > \rho, \;\; 3K_* > 2\mu \quad \text{or} \quad \rho_* < \rho, \;\; 2\mu_* < 3K. \end{aligned} \tag{81}$$

Clearly, inequalities (81) are more restrictive than those in (80), most notably in that they entail a relationship between the mass densities of the inclusion and the background. The principal reason for such distinction lies in the fact that [11] centered their analysis around the *elastostatic* ITP, deployed as an elliptic (and compact) perturbation of the featured (elastodynamic) ITP. Unfortunately, the weak formulation of the modified ITP employed in this study does not permit elastostatic analysis as it would formally require setting ρ and ρ_* in (11) and thus in (18) and (19) to zero, which both introduces unbounded terms and destroys the required H^1-structure of the quadratic form $\mathcal{A}(U,U)$. Despite this apparent limitation formulas (75) provide, for the first time, an objective set of sufficient conditions that ensure the well-posedness of the visco-elastodynamic ITP in a fairly general situation (where both the obstacle and the background solid can be heterogeneous, anisotropic, and dissipative) provided that the excitation frequency does not belong to (at most) countable spectrum of transmission eigenvalues.

7.1 Analytical examples

Assuming that either (63) or (64) holds, it is shown in Section 5 that the set of transmission eigenvalues characterizing ITP (9) is at most discrete. Except for the “elastic-viscoelastic” case examined in Theorem 6, however, it is not known whether this set is nonempty. For the ITP in acoustics, it was demonstrated in [17] that the transmission eigenvalues indeed exist for certain problem configurations. For completeness, this possibility is examined in the context of (visco-) elastic waves via two analytical examples.

Longitudinal waves in rods. Consider the interior transmission problem involving longitudinal waves in two thin prismatic rods having unit length and equal cross-sectional areas. In this setting, let $(E,E_*)\in\mathbb{C}^2$ and $(\rho,\rho_*)\in\mathbb{R}^2$ denote respectively the constant Young's moduli and mass densities of the two rods. One seeks a non-trivial displacement solution, (u,u_*), of the homogeneous ITP associated with frequency $\omega>0$ so that

$$\begin{aligned} &E_*\frac{d^2u_*}{dx^2}+\rho_*\omega^2u_*=0 && \text{in } [0,1], \\ &E\frac{d^2u}{dx^2}+\rho\,\omega^2u=0 && \text{in } [0,1], \\ &u_*(0)=u(0), \qquad u_*(1)=u(1), \\ &E_*\frac{du_*}{dx}(0)=E\frac{du}{dx}(0), \qquad E_*\frac{du_*}{dx}(1)=E\frac{du}{dx}(1). \end{aligned} \tag{82}$$

Clearly, the solution to (82) entails four unknown constants, computable from the algebraic system of equations whose determinant vanishes when ω is a transmission eigenvalue. To examine this possibility, one may adopt the inverse of the featured determinant, termed F_r, as an indicator function. On the basis of (82), on finds that

$$F_r = \left| \det \begin{pmatrix} 1 & 1 & -1 & -1 \\ e^{\mathrm{i}\frac{\omega}{c}} & e^{-\mathrm{i}\frac{\omega}{c}} & -e^{\mathrm{i}\frac{\omega}{c_*}} & -e^{-\mathrm{i}\frac{\omega}{c_*}} \\ \frac{E}{c} & -\frac{E}{c} & -\frac{E_*}{c_*} & \frac{E_*}{c_*} \\ \frac{E}{c}e^{\mathrm{i}\frac{\omega}{c}} & -\frac{E}{c}e^{-\mathrm{i}\frac{\omega}{c}} & -\frac{E_*}{c_*}e^{\mathrm{i}\frac{\omega}{c_*}} & \frac{E_*}{c_*}e^{-\mathrm{i}\frac{\omega}{c_*}} \end{pmatrix} \right|^{-1}, \tag{83}$$

where $c = \sqrt{E/\rho}$ and $c_* = \sqrt{E_*/\rho_*}$ denote the phase velocities in the two rods. The left panel in Fig. 3 plots F_r versus ω for the "elastic-elastic" case assuming $E_* = 2E \in \mathbb{R}$ and $\rho_* = 2\rho$, noting that the featured set of material parameters conforms with the one-dimensional variant of (63) which guarantees that the set of transmission eigenvalues is at most countable. From the display, one can clearly see the indication of transmission eigenvalues, spread uniformly along the frequency range of interest. As a complement to this result, the right panel in Fig. 3 plots F_r versus ω for the "elastic-viscoelastic" case which assumes $E_* = (2 + 0.1\mathrm{i})\mathrm{E} \in \mathbb{C}$ and $\rho_* = 2\rho$. Consistent with the claim of Theorem 6, the latter result indicates absence of transmission eigenvalues when E is real and E_* is complex-valued (note that the local maximum at $\omega = 0$, present in both diagrams, takes significantly smaller value than the truncated "dynamic" maxima in the left panel).

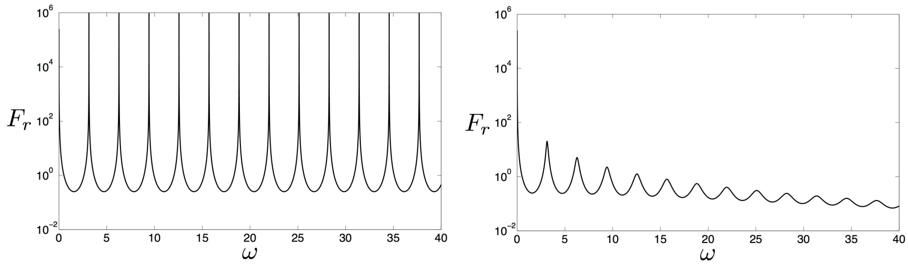

Fig. 3: Eigenvalue indicator F_r versus vibration frequency ω: "elastic-elastic" case, $(E, E_*) \in \mathbb{R}^2$ (left panel) and "elastic-viscoelastic" case, $(E, E_*) \in \mathbb{R} \times \mathbb{C}$ (right panel).

Oscillations of spheres. The second example deals with the ITP for two homogeneous and isotropic spheres of unit radius, characterized by the respective shear moduli $(\mu, \mu_*) \in \mathbb{C}^2$, Poisson's ratios $(\nu, \nu_*) \in \mathbb{R}^2$, and mass densities $(\rho, \rho_*) \in \mathbb{R}^2$. Once again, the transmission eigenvalues are associated with non-trivial solutions to the homogeneous ITP for which the two spheres share the Cauchy data on the boundary. Assuming that the two spheres are subjected respectively to constant radial pressures p and p_*, the induced (radial) boundary displacements u and u_* can be computed following [3] as

$$\begin{aligned} u_* &= \frac{p_*}{4\mu_*} \frac{Q_* \cos(Q_*) - \sin(Q_*)}{Q_* \cos(Q_*) - (1 - \alpha_*^2 Q_*^2) \sin(Q_*)}, \\ u &= \frac{p}{4\mu} \frac{Q \cos(Q) - \sin(Q)}{Q \cos(Q) - (1 - \alpha^2 Q^2) \sin(Q)}, \end{aligned} \tag{84}$$

where

$$\alpha^2 = \frac{1-\nu}{2-4\nu}, \qquad \alpha_*^2 = \frac{1-\nu_*}{2-4\nu_*}, \qquad Q^2 = \frac{\rho\,\omega^2}{4\mu\alpha^2}, \qquad Q_*^2 = \frac{\rho_*\omega^2}{4\mu_*\alpha_*^2}. \tag{85}$$

To develop an eigenvalue indicator function in the spirit of the previous example, one may assume that equality $p = p_*$ holds on the boundary, and define

$$F_s = \frac{|uu_*|}{\left|\frac{u}{p} - \frac{u_*}{p_*}\right|}, \tag{86}$$

as a quantity which becomes unbounded when ω is a transmission eigenvalue. As an illustration, the left panel in Fig. 4 plots F_s versus ω for the "elastic-elastic" case assuming $\mu_* = 2\mu \in \mathbb{R}$, $\nu_* = \nu = 1/8$ and $\rho_* = 2\rho$, while the right panel describes the corresponding "elastic-viscoelastic" situation where $\mu_* = (2+0.1\mathrm{i})\mu \in \mathbb{C}$. Similar to the previous example, the numerical results indicate the existence of transmission eigenvalues when both spheres are elastic, as well as their suppression when one of the two spheres is dissipative.

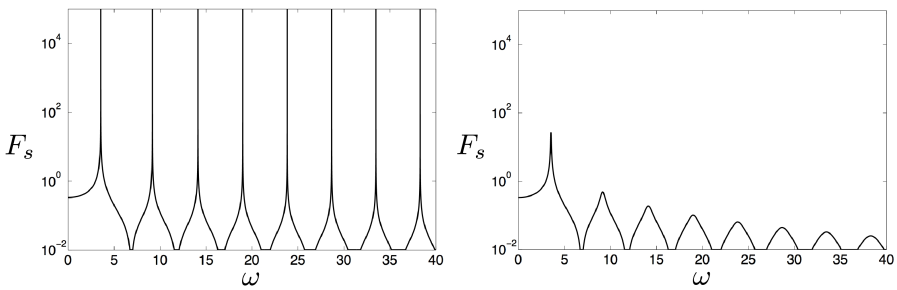

Fig. 4: Eigenvalue indicator F_s versus vibration frequency ω: "elastic-elastic" case, $(\mu,\mu_*) \in \mathbb{R}^2$ (left panel) and "elastic-viscoelastic" case, $(\mu,\mu_*) \in \mathbb{R}\times\mathbb{C}$ (right panel).

Viscoelastic-viscoelastic case. In the above examples, the focus was made on "conventional" ITP configurations where neither or either of the two bodies is dissipative. In light of the results in Section 6 where the analysis used to demonstrate the absence of transmission eigenvalues in the "elastic-viscoelastic" case failed to yield the same result for "viscoelastic-viscoelastic" (VV) configurations, it is of interest to examine the latter class of problems via the example of oscillating spheres. To ascertain whether transmission eigenvalues could indeed exist in the VV case, the spheres problem is approached from an alternative point of view, namely by fixing the vibration frequency at $\omega = \omega_{\rm o} \in \mathbb{R}$, and then seeking admissible sets of viscoelastic parameters for which $\omega_{\rm o}$ is a transmission eigenvalue. To this end, one may introduce an auxiliary set of material parameters $(\beta,\gamma) \in \mathbb{C}^2$ and $(\beta_*,\gamma_*) \in \mathbb{C}^2$ as

$$\beta = \mu\alpha^2, \qquad \gamma = \frac{\alpha^2}{\mu}, \qquad \beta_* = \mu_*\alpha_*^2, \qquad \gamma_* = \frac{\alpha_*^2}{\mu_*}. \tag{87}$$

From (85) and (87), one finds

$$Q^2 = \frac{\rho\omega^2}{4\beta}, \qquad Q_*^2 = \frac{\rho_*\omega^2}{4\beta_*},$$

which allows the boundary displacements in (84) to be rewritten as

$$u_* = \frac{p_*}{4}\left(\frac{\gamma_*}{\beta_*}\right)^{\frac{1}{2}} \frac{Q_*\cos(Q_*) - \sin(Q_*)}{Q_*\cos(Q_*) - [1-(\beta_*\gamma_*)^{\frac{1}{2}}Q_*^2]\sin(Q_*)},$$
$$u = \frac{p}{4}\left(\frac{\gamma}{\beta}\right)^{\frac{1}{2}} \frac{Q\cos(Q) - \sin(Q)}{Q\cos(Q) - [1-(\beta\gamma)^{\frac{1}{2}}Q^2]\sin(Q)}. \tag{88}$$

Given $\omega_0 \in \mathbb{R}$, $(\rho,\rho_*) \in \mathbb{R}^2$, and $(\beta,\beta_*,\gamma_*) \in (\mathbb{C}\backslash\mathbb{R})^3$, one is now in position to seek $\gamma \in \mathbb{C}\backslash\mathbb{R}$ such that $u = u_*$ and $p = p_*$. On the basis of (88), the explicit solution is given by

$$\gamma = \frac{\beta\Lambda^2(Q\cos(Q) - \sin(Q))^2}{[Q\cos(Q) - (1+\Lambda\beta Q^2)\sin(Q)]^2}, \tag{89}$$

where

$$\Lambda = \left(\frac{\gamma_*}{\beta_*}\right)^{\frac{1}{2}} \frac{Q_*\cos(Q_*) - \sin(Q_*)}{Q_*\cos(Q_*) - [1-(\beta_*\gamma_*)^{\frac{1}{2}}Q_*^2]\sin(Q_*)}. \tag{90}$$

In this setting, any relevant solution in terms of γ must also satisfy the conditions of physical admissibility in terms of the shear and bulk moduli

$$\mu = \left(\frac{\beta}{\gamma}\right)^{\frac{1}{2}}, \qquad K = 4\beta - \frac{4}{3}\left(\frac{\beta}{\gamma}\right)^{\frac{1}{2}},$$

which are subject to the ellipticity and thermomechanical stability requirements

$$\Re[\mu] > 0, \qquad \Im[\mu] \geqslant 0, \qquad \Re[K] > 0, \qquad \Im[K] \geqslant 0. \tag{91}$$

Despite the multitude of inequality constraints in (91), it is indeed possible to find an admissible solution (89) in terms of γ given ω_0, (ρ,ρ_*) and (β,β_*,γ_*) as shown in Table 2. For completeness, this result is accompanied by the variation of the eigenvalue indicator function (86) in Fig. 5, where F_s is plotted versus frequency for each of the three VV configurations highlighted in Table 2. From the display, it is seen that the three diagrams of F_s exhibit apparent "blow-off" behavior respectively at $\omega = 2, 10$ and 25 as expected. In unison, Table 2 and Fig. 5 provide a clear indication that the transmission eigenvalues may appear even in situations when both the obstacle and the background solid are viscoelastc i.e. dissipative - a finding that may be especially relevant in the application of inverse scattering theories to real-life problems (e.g. seismic imaging) were many materials are known to exhibit dissipative constitutive behavior.

Table 2: Oscillating spheres problem - VV configuration: numerical values of material parameters for which $\omega = \omega_0$ is a transmission eigenvalue.

ω_0	ρ	ρ_*	μ	μ_*	K	K_*	Config.
2	3	1.5	$8.833 + 1.214i$	$3.139 + 0.314i$	$12.22 + 0.781i$	$11.82 + 0.782i$	1
10	3	1.5	$4.157 + 1.684i$	$3.139 + 0.314i$	$26.46 + 0.155i$	$11.82 + 0.782i$	2
25	6	3.4	$173.6 + 4.320i$	$1.414 + 0.071i$	$368.5 + 52.24i$	$14.11 + 1.106i$	3

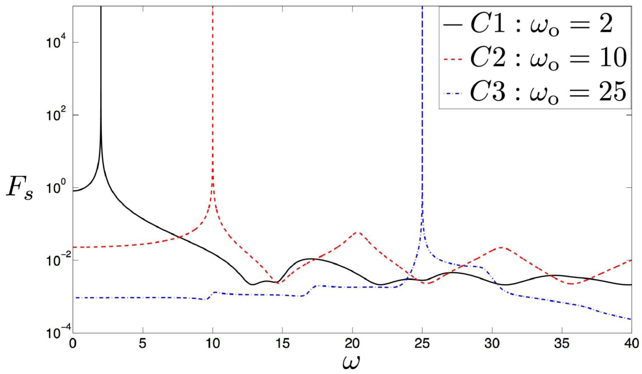

Fig. 5: Oscillating spheres problem - VV configuration: numerical manifestation of the transmission eigenvalues exposed in Table 2

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8 Conclusions

In this study the analysis of the interior transmission problem (ITP), that plays a critical role in a number of inverse scattering theories, is extended to enable the treatment of problems in piecewise-homogeneous, anisotropic, elastic and viscoelastic solids involving multiply-connected penetrable and impenetrable obstacles. Making recourse to a particular variational formulation, the Lax-Milgram theorem, and the compact perturbation argument, a set of sufficient conditions is established in terms of the elasticity and density parameters of the obstacle and the background solid that ensure the ellipticity of the ITP provided that the excitation frequency does not belong to (at most) countable set of transmission eigenvalues. It is further shown that this set is empty in situations when either the obstacle or the background solid are dissipative i.e. viscoelastic. When *both* the obstacle and the background are either elastic or viscoelastic, on the other hand, the same type of analysis fails to produce any further restrictions on the (countable) set of transmission eigenvalues. Given the counter-intuitive nature of such finding for the “viscoelastic-viscoelastic” (VV) case, the problem is further investigated via an energetic argument which shows that the homogeneous ITP involving VV configurations is not mechanically isolated from its surroundings in that it permits a non-zero energy influx into the system even though the prescribed excitation (given by the jump in Cauchy data between the two bodies) vanishes. A set of numerical results, computed for configurations that meet the sufficient “solvability” conditions, is included to illustrate the problem. Consistent with the underpinning analysis, the results indicate that the set of transmission values is indeed empty in the “elastic-viscoelastic” case, and countable for the “elastic-elastic” and VV configurations.

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